Numerical implementation of a hydro-mechanical model for partially saturated soils

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Abstract

In this paper, a rigorous numerical implementation scheme is introduced for coupled analysis of flow and deformation in variably saturated soils. The hydro-mechanical model is formulated incrementally within a three phase framework using mass and momentum balance equations and the effective stress approach. The stress-strain behaviour is described through an effective stress state based elasto-plastic constitutive model. Numerical implementation of the coupled flow-deformation equations together with the elasto-plastic constitutive model is demonstrated through the finite element approach. The governing equations are discretised spatially using the standard Galrekin method while the finite difference technique is employed for the discretisation of the time domain. The elasto-plastic constitutive equations are integrated using an explicit integration algorithm. A stress correction scheme is introduced with particular emphasis on improving accuracy, robustness and efficiency of the solution scheme. Performance of the model is investigated by comparing numerical predictions with experimental results for a range of loading paths in variably saturated soils.

1 INTRODUCTION

Unsaturated soils are routinely encountered in geotechnical engineering practice as compacted soils in roads, airfields, earth dams, embankments, *etc*. Other examples of unsaturated soils are: swelling clays, collapsing soils and residual soils. Despite the widespread occurrence of unsaturated soils throughout the world, the research into the behaviour of these soils has been relatively limited. The main purpose of this paper is to present a simple but efficient and robust numerical scheme for a fully coupled analysis of flow and deformation problems in variably saturated soils. The governing equations for the flow and deformation models are formulated within the context of the theory of mixtures and the hydro-mechanical coupling is achieved through the principle of effective stresses. Introduction of the effective stress enables description of unsaturated soil behaviour using the same framework as saturated soil mechanics. In addition, the elasto-plasticity model accounts for unsaturated behaviour through the stiffening effect of matric suction. The resulting elasto-plastic stress-strain relationships are integrated using an explicit algorithm with a yield surface drift correction scheme. Application of the model to variably saturated soils is demonstrated through numerical predictions of drying path and drained triaxial loading tests on variably saturated soils.

2 GOVERNING EQUATIONS

Unsaturated soils are three-phase porous media consisting of water, air and soil grains. Following the theory of mixtures, each phase is endowed with its own kinematics, mass and momentum. The governing equations for the flow of water and air are derived using the conservation equations of mass and momentum, while the deformation equations of the soil matrix are obtained satisfying the conditions of equilibrium, compatibility and consistency (Khalili et al. 2000, Loret and Khalili 2000). The incremental effective stress for unsaturated soils is given by (Khalili et al. 2004)

$$\sigma'_{ij} = \sigma_{ij}^{net} + \psi s \delta_{ij} \tag{1}$$

where a superimposed dot indicates rate of increment, $\sigma_{ij}^{net} = \sigma_{ij} - p_a \delta_{ij}$ is the net stress, σ_{ij} is the total stress, $s = p_a - p_w$ is the matric suction, p_w & p_a are pore water and pore air pressures, ψ is the incremental effective stress parameter, and δ_{ij} is Kronecker's delta.

The coupling between the flow and deformation models is established using the concept of effective stress and through the compatibility requirement of the volumetric deformations of the three phases. Applying Betti's reciprocal law (Khalili and Valliappan 1996, Khalili et al. 2000) to compute the volumetric deformations of the water and air phases, and combining them with the governing equations for the flow and deformation models results in a fully coupled hydro-mechanical model for variably saturated soils. The coupled equations for water flow, air flow and deformation of the soil matrix in a three phase unsaturated soil medium are given by (Habte 2006),

$$-\psi \frac{\partial \dot{u}_i}{\partial x_i} - a_{11}\dot{p}_w + \frac{\partial}{\partial x_i}\left(\frac{k_{wij}}{\mu_w}\frac{\partial p_w}{\partial x_j}\right) + a_{12}\dot{p}_a = 0 \tag{2}$$

$$-(1-\psi)\frac{\partial \dot{u}_i}{\partial x_i} + a_{21}\dot{p}_w + \frac{\partial}{\partial x_i}\left(\frac{k_{aij}}{\mu_a}\frac{\partial p_a}{\partial x_j}\right) - a_{22}\dot{p}_a = 0 \tag{3}$$

$$D^{ep}_{ijkl}\frac{\partial \dot{u}^2_{ij}}{\partial x_i \partial x_j} - \psi \frac{\partial \dot{p}_w}{\partial x_i} - (1-\psi)\frac{\partial \dot{p}_a}{\partial x_i} + F_i = 0 \tag{4}$$

where $a_{11} = a_{12} = a_{21} = c'_m - \psi^2/K$ and $a_{22} = a_{21} + n_a/K_a$, k_{wij} & k_{aij} are the permeability matrices of water and air, u_i is the deformation vector of the soil mass, D^{ep}_{ijkl} is the elasto-plastic stiffness matrix of the soil, x_i is the coordinate direction, F_i is the body force per unit volume, c'_m is compressibility of the water phase with respect to change in matric suction, K is the elastic bulk modulus of the soil mass, n_a is the volumetric fraction of the air phase, K_a is compressibility of air.

3 ELASTO-PLASTIC STRESS-STRAIN RELATIONSHIP

The stress-strain relationship of the soil mass is described using an effective stress based elastic-plastic constitutive model. Similar to saturated soils, the yield surface and plastic potential for unsaturated soils are expressed in terms of the effective stress. However, the plastic hardening modulus for unsaturated soils incorporates the hardening effect of matric suction (Loret and Khalili 2000). Incorporation of suction into the constitutive framework facilitates simulation of the complex phenomena observed in unsaturated soils, such as collapse during wetting and elastic swelling during drying.

Using matrix-vector notation, the incremental stress-strain equation is given by

$$\dot{\boldsymbol{\sigma}}' = \mathbf{D}^{\mathrm{e}}\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} \tag{5}$$

where $\dot{\boldsymbol{\sigma}}'$ is the incremental effective stress, $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ is the incremental elastic strain, $\mathbf{D}^{\mathrm{e}}$ is the elastic compliance matrix. Following plasticity theory, the plastic strains are obtained by combining the plastic flow rule with the consistency condition on the yield surface.

$$\dot{\boldsymbol{\varepsilon}}^{p} = \frac{1}{h}\mathbf{n}^{\mathrm{T}}\mathbf{m}\dot{\boldsymbol{\sigma}}' \tag{6}$$

Where $\dot{\boldsymbol{\varepsilon}}^p$ is the incremental plastic strain, h is the plastic modulus, $\mathbf{n}$ is the unit normal to the yield surface and $\mathbf{m}$ is the unit direction of plastic flow at $\boldsymbol{\sigma}'$. Noting that the total strain increment is given by the sum of the elastic and plastic components ($\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$) and substituting equation (6) into the basic stress-strain equation (5), the elasto-plastic stress-strain relationship for unsaturated soils becomes

$$\dot{\boldsymbol{\sigma}}' = \left(\mathbf{D}^{\mathrm{e}} - \frac{\mathbf{D}^e \mathbf{m} \mathbf{n}^{\mathrm{T}} \mathbf{D}^e}{h + \mathbf{n}^{\mathrm{T}} \mathbf{D}^e \mathbf{m}} \right) \dot{\boldsymbol{\varepsilon}} = \mathbf{D}^{\mathrm{ep}} \, \dot{\boldsymbol{\varepsilon}} \tag{7}$$

in which $\mathbf{D}^{ep}$ is the elasto-plastic stress-strain matrix.

4 NUMERICAL IMPLEMENTATION

4.1 Finite Element Formulation

The numerical solution to the coupled governing equations presented above requires discretisation of the problem in space and time domains. The discretisation in space is done using a standard finite element procedure. Applying Galerkin's residual method to the weak form of the governing equations (2), (3) and (4) yields the following system of discrete equations in terms of the element property matrices and nodal variables.

$$[\mathbf{K}]\{\dot{\mathbf{u}}\} - \psi[\mathbf{C}]\{\dot{\mathbf{p}}_w\} - (1-\psi)[\mathbf{C}]\{\dot{\mathbf{p}}_a\} = \{\dot{\mathbf{P}}\}$$

$$-\psi[\mathbf{C}]^{\mathrm{T}}\{\dot{\mathbf{u}}\} - a_{11}[\mathbf{M}]\{\dot{\mathbf{p}}_w\} - [\mathbf{H}_w]\{\mathbf{p}_w\} + a_{12}[\mathbf{M}]\{\dot{\mathbf{p}}_a\} = \{\mathbf{Q}_w\} \tag{8}$$

$$-(1-\psi)[\mathbf{C}]^{\mathrm{T}}\{\dot{\mathbf{u}}\} + a_{21}[\mathbf{M}]\{\dot{\mathbf{p}}_w\} - [\mathbf{H}_a]\{\mathbf{p}_a\} - a_{22}[\mathbf{M}]\{\dot{\mathbf{p}}_a\} = \{\mathbf{Q}_a\}$$

in which $[\mathbf{K}]$ is the element stiffness matrix, $[\mathbf{C}]$ is the coupling matrix, $[\mathbf{M}]$ is the mass matrix, $[\mathbf{H}_w]$ and $[\mathbf{H}_a]$ are flow matrices corresponding to the permeabilities of the water and air phases respectively, $\{\mathbf{u}\}$ is the vector of nodal displacements, $\{\mathbf{p}_w\}$ is the vector of nodal pore water pressures, $\{\mathbf{p}_a\}$ is the vector of nodal pore air pressures, $\{\mathbf{P}\}$ is the vector of nodal forces, $\{\mathbf{Q}_w\}$ and $\{\mathbf{Q}_a\}$ are vectors of nodal fluxes of the water and air flows respectively.

Applying the finite difference approach to the primary variables in the discretized equations (8) yields the following linearized form of the governing equations,

$$[\mathbf{K}]\{\Delta\mathbf{u}\} - \psi[\mathbf{C}]\{\Delta\mathbf{p}_w\} - (1-\psi)[\mathbf{C}]\{\Delta\mathbf{p}_a\} = \{\Delta\mathbf{R}\}$$

$$-\psi[\mathbf{C}]^{\mathrm{T}}\{\Delta\mathbf{u}\} - (a_{11}[\mathbf{M}] + \beta\Delta t[\mathbf{H}_w])\{\Delta\mathbf{p}_w\} + a_{12}[\mathbf{M}]\{\Delta\mathbf{p}_a\} = \left[(1-\beta)\{\mathbf{Q}_w\}_t + \beta\{\mathbf{Q}_w\}_{t+\Delta t} + [\mathbf{H}_w]\{\mathbf{p}_w\}_t\right]\Delta t \tag{9}$$

$$-(1-\psi)[\mathbf{C}]^{\mathrm{T}}\{\Delta\mathbf{u}\} + a_{21}[\mathbf{M}]\{\Delta\mathbf{p}_w\} - (a_{22}[\mathbf{M}] + \beta\Delta t[\mathbf{H}_a])\{\Delta\mathbf{p}_a\} = \left[(1-\beta)\{\mathbf{Q}_a\}_t + \beta\{\mathbf{Q}_a\}_{t+\Delta t} + [\mathbf{H}_a]\{\mathbf{p}_a\}_t\right]\Delta t$$

where t is the time and β is a parameter controlling the type of interpolation.

4.2 Stress Integration Scheme

An important aspect of the numerical solution to the elasto-plastic constitutive equations developed in Section 3 above is implementation of an efficient integration scheme for computation of stresses and hardening parameters as accurately as possible. Explicit rather than implicit integration schemes are preferred here as they are advantageous in regards to efficiency, robustness and accuracy for complex constitutive equations. Yield surface correction schemes are also incorporated in the solution procedure to improve accuracy by controlling the tendency of the computed stress point to drift from the yield surface.

The Euler forward scheme, also known as the tangential stiffness method, is a first order algorithm in which the constitutive equations are integrated directly using the elasto-plastic stiffness matrix computed at a previously known stress point. Using Euler's forward scheme, the stress at the next increment ($\boldsymbol{\sigma}'_{t+\Delta t}$) is calculated from

$$\boldsymbol{\sigma}'_{t+\Delta t} = \boldsymbol{\sigma}'_t + \Delta\boldsymbol{\sigma}' \tag{10}$$

in which Δt is the step size and $\boldsymbol{\sigma}'_t$ is the current stress. $\Delta\boldsymbol{\sigma}' = \mathbf{D}^{\mathrm{ep}}\Delta\boldsymbol{\varepsilon}$ is computed by direct integration of equation (7) with $\Delta\boldsymbol{\varepsilon}$ being the imposed incremental strain.

4.3 Correction of Yield Surface Drift

In plasticity theory, yield surface drift is a condition where the computed stresses fail to satisfy the yield condition on the updated yield surface. The main source of this discrepancy is the assumption of constant elasto-plastic stress-strain matrix over the imposed strain increment although the actual elasto-plastic matrix varies continuously as the stress changes. Due to this, the predicted state of stress at the end of an increment may not lie on the current yield surface even for very small step sizes. The amount of deviation from the yield surface depends on the accuracy of the integration scheme and the nonlinearity of the constitutive relation. As any deviation of the stress point from the yield surface has a cumulative effect on subsequent computations, stresses must always be corrected back to the current yield surface at the end of each increment.

The consistent correction scheme implemented here involves applying corrections to both the stresses and the hardening parameter by updating both the elastic and plastic strains using the equations of plasticity and Taylor's series expansion of the yield condition. Based on theoretical, accuracy and efficiency considerations, the consistent correction scheme is considered to be the most reliable method of restoring stresses to the yield surface. The consistent drift correction scheme is given by (Sloan et al. 2001)

$$\delta\boldsymbol{\sigma}' = -\delta\Lambda\mathbf{D}^{\mathrm{e}}\mathbf{m} \tag{11}$$

in which $\delta\boldsymbol{\sigma}'$ is the stress correction, $\delta\Lambda$ is scalar multiplier and is computed using

$$\delta\Lambda = \frac{f_{tr}}{h + \mathbf{n}^{\mathrm{T}}\mathbf{D}^{e}\mathbf{m}} \tag{12}$$

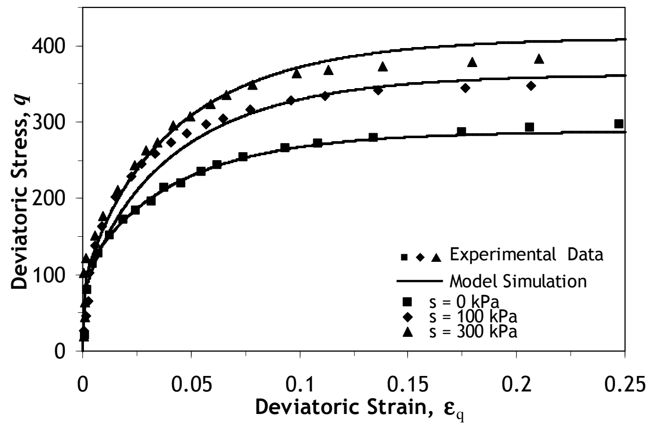
where f_{tr} is the yield value at the trial stress point. The correction is repeated iteratively until f_{tr} becomes less than a pre-specified tolerance limit.

5 APPLICATION

Performance of the proposed modelling framework is investigated by comparing numerical simulations with experimental data for drained triaxial loading and drying path (desaturation) tests on variably saturated soil samples. The experimental results presented below were taken from the series of tests conducted on laboratory compacted samples of silt from the Bourke region of New South Wales by Uchaipichat (2005). All simulations were obtained using a bounding surface plasticity theory (Khalili et al. 2005). Details of the model utilised and the material parameters adopted may be found in Habte (2006).

5.1 Drained Triaxial Tests

The triaxial tests considered here are drained compression tests at constant suction. Three sets of test data with a cell pressure of 150 kPa and suction values of 0, 100 and 300 kPa are shown below. All the samples were preconsolidated to an isotropic preconsolidation stress of 200 kPa. The results of model simulation are shown in Figure 1 using deviatoric stress – deviatoric strain and volumetric stress – deviatoric strain plots. Model predictions of the deviatoric and volumetric responses matched the experimental data reasonably well. These plots showed higher strength and larger volumetric deformation for tests at higher suction values. The higher strength was due to the increase in the yield stress as a result of suction hardening, whereas the larger deformation was due to the increase in the effective stress as a result of suction increase.



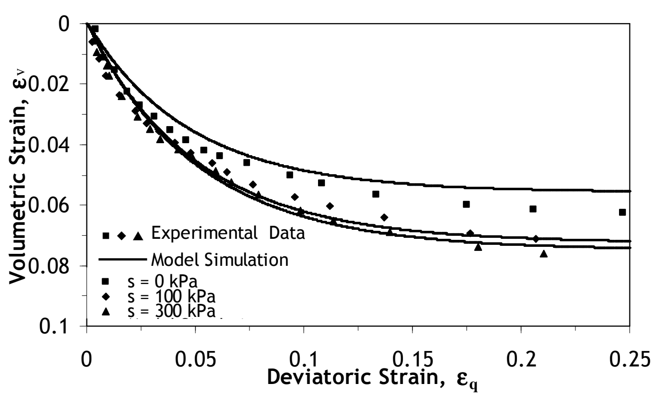


Figure 1: Comparison of model simulation and experimental data for triaxial compression tests on Bourke silt at a cell pressure of 150 kPa.

5.2 Drying Path Tests

The drying path (desaturation) tests were conducted by increasing the matric suction at constant net stress. The samples were initially saturated, isotropically preconsolidated to 200 kPa and unloaded to net stress values of 50, 100, 150 and 200 kPa. Matric suction was applied incrementally whilst monitoring the corresponding volume change in the sample. Comparison between results of model simulation and experimental data are shown in Figure 2 using volume change - matric suction plots. The figure shows the model simulations capture the soil response with good accuracy. Three of the drying tests were conducted on overconsolidated samples and hence resulted in elastic response. The fourth drying path test at a mean net stress of 200 kPa was performed on a normally consolidated sample, and the model successfully captured the transition from elasto-plastic to elastic behaviour during desaturation of the sample.

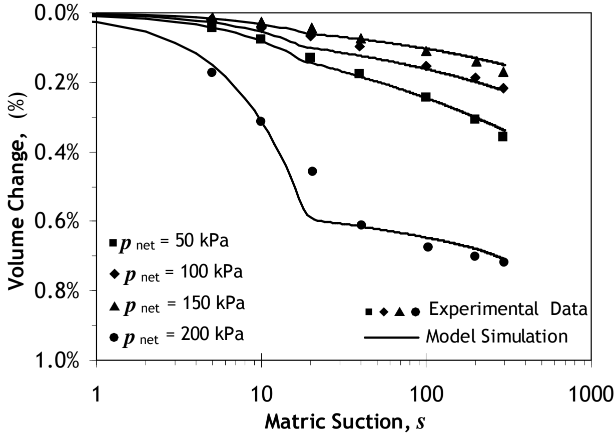


Figure 2: Comparison of model simulation and experimental data for drying path tests on Bourke silt

6 CONCLUSION

A fully coupled hydro-mechanical model for describing the flow and deformation behaviour of partially saturated soils is presented. The model is formulated incrementally using the effective stress principle. The governing equations for the flow model are formulated using mass and momentum balance equations with in the context of theory of mixtures. The constitutive response of unsaturated states is taken into account through the hardening effect of matric suction. Solution to the governing equations was obtained using the finite element and finite difference procedures. The stress-strain constitutive equations are integrated using an explicit integration algorithm with a yield surface drift correction scheme. Application of the model was demonstrated using typical triaxial loading and desaturation tests on normally and overconsolidated variably saturated samples of Bourke silt. In all the tests considered, performance of the model simulations in reproducing the experimentally observed response of unsaturated soils was satisfactory.

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