A general three dimensional bounding surface model for sand

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ABSTRACT

In this paper, a study of the constitutive modelling of soils in the three dimensional stress space is presented. The concept of generalised shear stress ratio is utilised to represent the mobilisation of the friction resistance of soil in the general stress state. A plastic potential, which can capture reliably the direction of plastic strain increment for three dimensional states, is proposed. A general bounding surface model is thus formulated based on a conventional two dimensional model. The model is verified against experimental data.

1 INTRODUCTION

In geotechnical engineering practice, soil states such as those in slope stability, tunnelling and excavation are more often than not in the general three dimensional state, usually involving shear stresses. However, most of the existing models are developed in two dimensional settings describing the stress and strain relationship of soils that are obtained from conventional triaxial tests. It has been widely reported that both the stiffness and the peak strength of sands are influenced by the intermediate stress. Indeed, the application of a two dimensional model for numerical analysis of geotechnical problems may lead to significant divergence from the "true" performance or wrong answers (Potts and Zdravkovic, 1999).

In this paper, a three dimensional elastoplastic bounding surface model for sand is formulated. A generalized shear stress ratio, derived from the critical state strength of a soil, is introduced as a mapping quantity for the mobilisation of the friction resistance of soil in the general stress state. A plastic potential, which can capture reliably the direction of plastic strain increment for three dimensional states, is proposed. The model is verified against experimental data where soil specimens are tested in the general stress space.

Compact tensorial notation is used throughout. Bold face letters indicate tensors, matrices and vectors. Symbols '·' and ':' between tensors of various orders denote inner product with single and double contraction, respectively. The dyadic product between two tensors is indicated with '⊗'. 'tr' denotes the tracer operator; e.g. for second order symmetric tensor $\boldsymbol{\sigma}$, $tr\,\boldsymbol{\sigma} = \boldsymbol{\delta} : \boldsymbol{\sigma}$, where $\boldsymbol{\delta}$ is the second order identity tensor.

The definitions of the general mean effective stress p', the generalized shear stress q, the volumetric strain increment $\dot{\varepsilon}_v$, and the distortional strain increment $\dot{\varepsilon}_d$ are given below, where σ'_1, σ'_2 and σ'_3 are the three principal stresses, and ε_1, ε_2 and ε_3 are the three principal strains.

$$p' = \frac{1}{3}(\sigma'_1 + \sigma'_2 + \sigma'_3) = \frac{\mathrm{tr}\boldsymbol{\sigma}'}{3} \qquad \text{and} \qquad q = \eta p' \tag{1}$$

$$\dot{\varepsilon}_v = (\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3) = \mathrm{tr}\dot{\varepsilon} \qquad \text{and} \qquad \dot{\varepsilon}_d = \frac{\sqrt{6}}{3}\left[\left(\dot{\boldsymbol{\varepsilon}} - \frac{1}{3}\dot{\varepsilon}_v\boldsymbol{\delta}\right) : \left(\dot{\boldsymbol{\varepsilon}} - \frac{1}{3}\dot{\varepsilon}_v\boldsymbol{\delta}\right)\right]^{\frac{1}{2}} \tag{2}$$

η is the generalized shear stress ratio, determined based on the form of the critical failure states in the π plane.

2 GENERAL THREE DIMENSIONAL BOUNDING SURFACE MODEL

2.1 Critical sate for sand and the generalized shear stress ratio

The proposed model is based on a generalized shear stress ratio for geomaterials based on their critical state strength. The new general shear stress ratio η (hereafter referred to as the present criterion) makes a simple and smooth transition between the von Mises criterion and the Matsuoka-Nakai criterion (1982) as follows,

$$\eta(\boldsymbol{\sigma}', m) = \sqrt{m\eta_{MN}^2(\boldsymbol{\sigma}') + (1-m)\eta_{vM}^2(\boldsymbol{\sigma}')} \tag{3}$$

where *m* is a material constant. η_{MN} and η_{vM} are the general shear stress ratios corresponding to the Matsuoka-Nakai criterion and the von Mises criterion respectively. They are given as

$$\eta_{vM}(\boldsymbol{\sigma}') = \frac{3\sqrt{6}}{2}\frac{\sqrt{(\boldsymbol{\sigma}' - p'\boldsymbol{\delta}):(\boldsymbol{\sigma}' - p'\boldsymbol{\delta})}}{\mathrm{tr}\boldsymbol{\sigma}'} \tag{4}$$

$$\eta_{MN}(\boldsymbol{\sigma}') = \sqrt{\frac{\mathrm{tr}\boldsymbol{\sigma}'\left[(\mathrm{tr}\boldsymbol{\sigma}')^2 - (\boldsymbol{\sigma}':\boldsymbol{\sigma}')\right]}{18J_3(\boldsymbol{\sigma}')} - 1} \tag{5}$$

where J_3 is the third stress invariant and is defined as

$$J_3(\boldsymbol{\sigma}') = \sigma_1'\sigma_2'\sigma_3' = \frac{1}{6}\left[2\mathrm{tr}(\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}') - 3\mathrm{tr}\boldsymbol{\sigma}'(\boldsymbol{\sigma}':\boldsymbol{\sigma}') + (\mathrm{tr}\boldsymbol{\sigma}')^3\right] \tag{6}$$

The proposed generalized shear stress ratio is simple and straightforward. Some of the deficiencies of the existing failure criterions have been removed. (1) The criterion has the capacity to describe accurately the variation of the strengths of geomaterials from the Mohr-Coulomb criterion to the von Mises criterion. (2) It does not allow the occurrence of tensile stress state except when the criterion is coincided with the von Mises criterion. (3) The surface in the π plane is always convex as the surfaces for both Matsuoka-Nakai criterion and von Mises criterion are unconditionally convex in the plane. Consequently, the convexity of the yield surface, formed based on the proposed general shear stress ratio, can be guaranteed if the corresponding yield surface in the two dimensional stress is convex.

The selection of a suitable mapping quantity for the mobilization of the friction resistance in the general stress state is an essential part of formulating a three dimensional model or generalising a two dimensional model. The general shear stress ratio proposed can be employed as an ideal mapping quantity to represent the mobilisation of the friction resistance of soil in the general stress state. It can be seen that the generalised stress ratio is independent of the Lode's angle and the value of the critical state stress ratio is independent of the stress states. Consequently, constitutive equations for the general stress and strain states can be formulated in a simple manner than those with critical state strength of sand dependent on Lode's angle (Gajo and Muir Wood, 1999).

2.2 Critical state line and limiting isotropic compression line

The critical state is an ultimate state for soils under shearing, and the critical state line (CSL) in the specific volume and the mean effective stress plane, i.e., the $v \sim \ln p'$ plane, is widely reported as material intrinsic properties. After reviewing several experimental data over a wide range of stresses, Russell and Khalili (2004) suggested the critical state of granular soils in the $v \sim \ln p'$ plane can be represented using a three segmented line.

Besides the critical state line, elastoplastic constitutive modelling requires definition of a limiting isotropic compression line (LICL), in order to formulate hardening rule for evolution of the yield/bounding surface. Similar to the critical state, LICL is a reference line in the $v \sim \ln p'$ plane where the stress state approaches with increasing isotropic compression. In the present investigation, the isotropic compression line is taken as parallel to the critical state line and at a constant shift along the κ line from the CSL in the $v \sim \ln p'$ plane.

2.3 Stress strain relationship

The material behaviour is assumed isotropic and rate independent in both elastic and elastoplastic responses. Thus the incremental stress and strain relationship can be expressed as

$$\dot{\boldsymbol{\sigma}}' = \left(\mathbf{D}^{\mathrm{e}} - \frac{\mathbf{D}^{\mathrm{e}} : \mathbf{m} \otimes \mathbf{n} : \mathbf{D}^{\mathrm{e}}}{h + \mathbf{n} : \mathbf{D}^{\mathrm{e}} : \mathbf{m}} \right) : \dot{\boldsymbol{\varepsilon}} \tag{7}$$

where $\mathbf{n}$ is the unit tensor normal to the loading surface at the current stress state $\boldsymbol{\sigma}'$, $\mathbf{m}$ is the unit direction of plastic flow at $\boldsymbol{\sigma}'$, and h is the hardening modulus. $\mathbf{D}^e$ is a fourth order tensor for the elastic property, which is well documented and is not discussed here.

2.4 Plastic behaviour

The plastic behaviour is captured using the bounding surface theory (Dafalias and Hermann, 1980). A brief summary is given here, and the details can be seen in Khalili et al (2007).

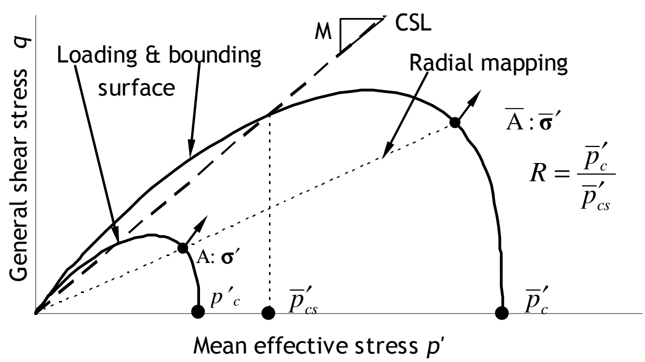

Figure 1: Bounding surface, loading surface and mapping rule for first loading

2.4.1 Bounding surface, loading surface, and the mapping rule

Based on the bounding surface proposed by Khalili et al (2005) for soil in axisymmetrical stress states, the following equation for the bounding surface of a general stress states is obtained with the proposed general shear stress ratio, e.g., equation (3),

$$F(\bar{p}', \bar{q}, \bar{p}'_c) = \frac{\bar{\eta}\,\mathrm{tr}\bar{\boldsymbol{\sigma}}'}{3} - \frac{M\,\mathrm{tr}\bar{\boldsymbol{\sigma}}'}{3} \left\{ \left(\frac{1}{\ln R} \right) \ln \left[\frac{3\bar{p}'_c}{\mathrm{tr}\bar{\boldsymbol{\sigma}}'} \right] \right\}^{1/N} = 0 \tag{8}$$

In equations, a superimposed bar indicates quantities associated with the bounding surface. A sketch of the bounding surface is shown in Figure 1. The parameter $\bar{p}'_c$ is the size of the bounding surface. Material constant *R* represents the ratio between $\bar{p}'_c$ and the value of $\bar{p}'$ at the intercept of the surface with the CSL in the $q \sim p'$ plane. Material constant *N* controls the curvature of the surface. *M* represents the critical state strength and is a material constant. *M* is the value of the shear stress ratio at a critical state.

For first time loading, from the state of zero stress, the loading and bounding surfaces are assumed of the same shape and homologous about the origin of stresses (Figure 1). In this case, the function for the loading surface takes the following form

$$f(\boldsymbol{\sigma}', p_c') = \frac{\eta \mathrm{tr}\boldsymbol{\sigma}'}{3} - \frac{M \mathrm{tr}\boldsymbol{\sigma}'}{3}\left\{\left(\frac{1}{\ln R}\right)\ln\left[\frac{3p_c'}{\mathrm{tr}\boldsymbol{\sigma}'}\right]\right\}^{1/N} = 0 \tag{9}$$

where p_c' is an isotropic hardening parameter controlling the size of the loading surface.

For a stress state $\boldsymbol{\sigma}'$ on the loading surface, the corresponding image stress state $\overline{\boldsymbol{\sigma}}'$ on the bounding surface is determined by a mapping rule, the unit normal at the two states is identical. Thus for first loading, radial mapping rule is valid (Figure 1).

2.4.2 Plastic potential

The plastic potential is derived based on Rowe (1962) dilatancy relationship with some modification. The plastic potential, which is written in terms of the image stress state $\overline{\boldsymbol{\sigma}}'$, is found as follows.

$$g(\overline{\boldsymbol{\sigma}}', \overline{p}_o') = \overline{\overline{t}}\left[\eta(\overline{\boldsymbol{\sigma}}', m_1)\overline{p}' - \frac{AM\overline{p}'}{A-1} - \frac{\overline{p}'}{A-1}\left(\frac{\overline{p}'}{\overline{p}_o'}\right)^{A-1}\right] \tag{10}$$

A is a material constant. The proposed plastic potential has the capacity to capture accurately the direction of the plastic strain increment because the shape of the plastic potential in the π plane can be different from the failure surface via the introduction of a new shape parameter, m_1, i.e.,

$$\eta(\overline{\boldsymbol{\sigma}}', m_1) = \sqrt{m_1 \eta_{MN}^2(\overline{\boldsymbol{\sigma}}') + (1 - m_1)\eta_{vM}^2(\overline{\boldsymbol{\sigma}}')} \tag{11}$$

2.4.3 Hardening modulus

The plastic hardening modulus h is divided into h_b, the modulus at $\overline{\boldsymbol{\sigma}}'$ on the bounding surface, and h_f, the modulus reflecting the closeness between the bounding surface and the loading surface. The determination of the two parts of the hardening moduli are in consistent with the previous study and can be found in the paper by Khalili et al (2005).

3 PERFORMANCE OF THE MODEL

In this section, the proposed bounding surface model is employed to simulate the behaviour of sand for loading in the general stress space. A set of true triaxial tests performed on Fuji sand is considered for simulation (Yamada, 1979). The stress path for testing is linear in the π plane, as shown in the insert in Fig. 2a. Tests are simulated for stress paths θ varying from 0° to 60°, where the intermediate principal stress σ'_2 varies from being equal to the minimum principal stress to being equal to the maximum principal stress. Two sets of simulations are made with different definitions of the general shear stress ratio and the corresponding the general shear stress. They are the general shear stress ratio defined according to von Mises criterion (i.e., $m = 0$ for the present criterion), and that according to the present criterion. Values of the model parameters identified for this soil are listed in Table 1.

Table 1: Values of model parameters

κ	ν	ϕ_{cs}	N	R	k	A	CSL
0.01	0.25	39°	1.2	2	2	0.7	$\upsilon = 2.42 - 0.11\ln p'$
η by von Mises criterion: $m = m_1 = 0$, $k_m = 0.65$.					η by the present criterion: $m = m_1 = 0.65$, $k_m = 0.95$.		

The comparison between the model simulations and the experimental data for tests is shown in Figures 2, 3 and 4 for the principal strains and the volumetric deformation. In these figures, experimental data are represented by various open symbols, and simulations are represented by solid lines. The general shear stress *q* corresponding to von Mises criterion is used in the presentation of the shear stress as this stress parameter is familiar to most readers and it represents the magnitude of the deviation of the stress state from the isotropic stress state.

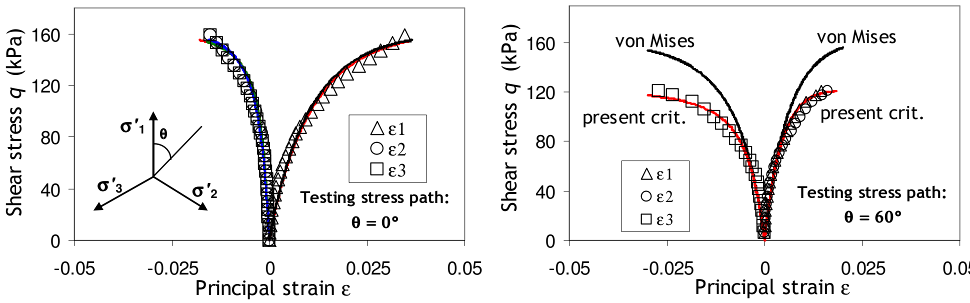

Figure 2 Comparison of model simulation and experimental data for tests with θ=0° and θ=60°

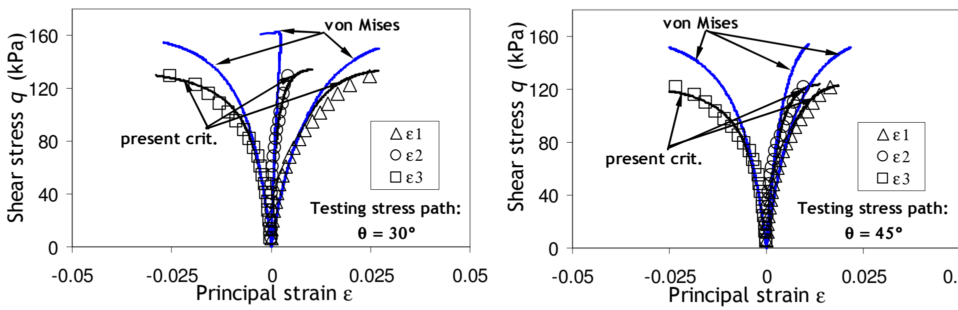

Figure 3 Comparison of model simulation and experimental data for tests with θ=30° and θ=45°

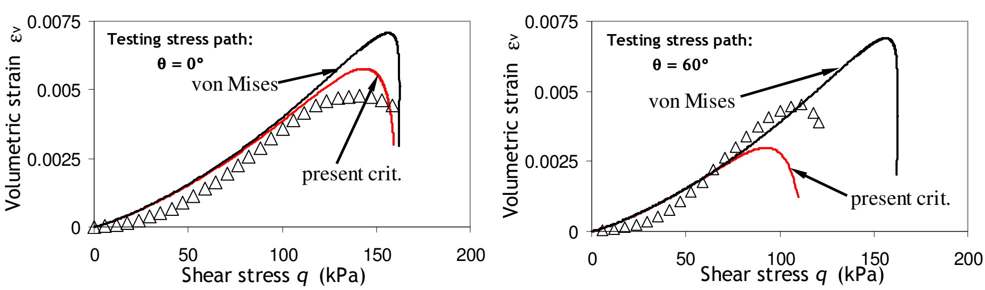

Figure 4 Volumetric deformation of Fuji sand for tests with θ=0° and θ=60°

It may be seen from Figure 2 that the two versions of the generalized model give highly satisfactory simulation of the soil behaviour for θ =0°, and the performance of the models is essentially the same. However, the performance of the two models varies greatly for the test with θ =60°. As was expected for models with von Mises criterion (Potts et al, 1999), tensile minimum principal stress is predicted even before soil failed under conventional triaxial extension condition. In the simulation, a value of σ'_3 = -10 kPa was predicted for the sand at peak strength state. This naturally leads to a high overestimation of the stiffness as well as the strength of the soil under conventional triaxial extension condition. It may also be seen from the comparison shown in Figures 2 and 3 that the von Mises model has a tendency of overestimating the stiffness and strength of the soil when the model parameters are determined from conventional triaxial compression tests and the model is used to represent soil behaviour that diverges from conventional triaxial compression stress states. Similarly, the model will have a tendency of underestimating the stiffness and strength of the soil when the model parameters are determined from conventional triaxial extension tests and the model is used to represent soil behaviour that diverges from the stress states. It can also be seen from the comparison for both the principal strains and the volumetric strain, the model with the present criterion gives satisfactory simulation of soil behaviour.

It is proposed to derive a generalized shear stress ratio, based on the adopted failure criterion, as a mapping quantity for representing the degree of the mobilization of the friction resistance of the soil. From the performance of the two versions of the general model, it may be concluded that the proposed method for formulating general constitutive models is rational.

4 CONCLUSION

A bounding surface model of sands for loading in the general stress and strain space is developed. A generalized shear stress ratio, independent of the Lode's angle, is proposed as a mapping quantity for representing the mobilization of the friction resistance of soil in the general stress state. Consequently, constitutive equations for the general stress and strain states can be formulated in a simple manner and the convexity of the loading surface is guaranteed. The model is used to simulate experimental data where soil specimens were tested in the general stress space. It is seen that the model has successfully captured the deformation of sand for loading in the general stress space by consistent values of model parameters.

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