Two Dimensional Simulation of Soil Moisture around a Leaking Water Pipe adjacent to a Concrete Slab

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Summary A method of predicting transient moisture distributions in soils beneath covered areas has been extended to examine the effects of a leaking water pipe adjacent to a concrete slab. The two dimensional unsaturated flow behaviour is predicted using solutions to the fully linearised Richards equation. Laplace and Fourier transforms are exploited to reduce the governing partial differential equation to an ordinary differential equation which can be solved analytically for a horizontal soil layer. Solutions for a layered soil profile are possible using a Finite Layer solution approach. The presence of a slab-on-ground is modelled by prescribing a zero flux segment at the upper boundary. Climatic effects such as rainfall and evaporation events are modelled using Type 3 boundary conditions, which regulate inflow and outflow as a linear function of the surface moisture content. The method of applying a Type 3 boundary condition in a transform solution formulation is described briefly. The presence of a leaking pipe adjacent to the edge of the slab is modelled by prescribing a source segment at a given depth. An example is provided.

1. INTRODUCTION

This paper describes work carried out as part of an integrated research project on reactive soils being undertaken at the University of Newcastle. This project is investigating reactive soil phenomena associated with lightly loaded residential structures. Its aims are to gain a better understanding of moisture and volume changes beneath covered areas on clay soil sites so that the currently employed methods of prediction and design can be improved. There are several aspects to the project including field trials, laboratory and field investigations and theoretical modelling.

The work presented here stems from the theoretical component. It involves the development of a computer program to predict unsaturated moisture distributions in layered soil profiles beneath covered areas, such as concrete building slabs. The long term aim is to verify the program using field data obtained in other components of the project, and then to couple the moisture change predictions with volume change and soil–foundation interaction models, to yield estimates of foundation response.

Although the effects of partial saturation are not often considered in everyday geotechnical practice, research into partially saturated soils has been, and continues to be, widespread in fields ranging from soil mechanics to agriculture. Computer codes already exist which use finite element techniques to solve the partially saturated flow problem in 2 and 3 dimensions. While the results of these codes are typically good, solutions in multiple dimensions are computationally laborious, especially for long term analyses. As one of the aims of this project is to improve the methods of prediction and design used in everyday practice, it desirable that the predictive code be fast and robust on typical office computers. To this end, a code using a new analytical approach is being developed. The properties of the Laplace and Fourier transforms are exploited to give stable and accurate solutions with a considerable reduction in computational cost.

This paper outlines the solution of the 2-dimensional moisture distribution problem, extended to consider the effects of a moisture source at some depth adjacent to the edge of a slab. It describes the application of a Type 3 boundary condition to simulate the case of a constant moisture content prescribed at the upper surface of a layer. Typical solutions are also presented.

2. THEORETICAL APPROACH

Pullan (1990) presents a comprehensive overview of the substantial body of unsaturated flow theory which has been developed to date. Only the major assumptions which are pertinent to the adopted form of the governing equation, will be discussed here.

Moisture flow is generally described by Darcy's law, which can be expressed mathematically as

$$U = -K.\nabla\Phi \tag{1}$$

where U is the flow velocity vector, K is the hydraulic conductivity, Φ is the total potential and ∇ is the gradient operator. In the case of unsaturated flow, the hydraulic conductivity, K, is a function of the moisture content ϑ, and the total potential has two components: the gravitational potential, z, and moisture potential Ψ (also a function of the moisture content).

Darcy's law (expressed in terms of a hydraulic conductivity function) can be combined with the total potential, and the principle of fluid flow continuity, to yield (2). This is traditionally referred to as the Richards equation.

$$\frac{\partial\vartheta}{\partial t} = \nabla.(D\nabla\vartheta) \qquad (2)$$

In this instance, Richards equation is expressed in terms of the soil diffusivity, $D(\vartheta)$, as defined in equation (3).

$$D(\vartheta) = K(\vartheta)\frac{\partial\Psi}{\partial\vartheta} \qquad (3)$$

To obtain a form of the Richards equation which can be readily solved using transform methods, the following assumptions and algebraic manipulations are applied.

- Application of a Kirchhoff transformation in which ϑ is transformed into a new variable θ by means of equation (4),

$$\theta = \int_{\Psi_1}^{\Psi} K(\Psi)\, d\Psi = \int_{\vartheta_1}^{\vartheta} D(\vartheta)\, d\vartheta \qquad (4)$$

in which ϑ_1 and $\Psi_1 = \Psi(\vartheta_1)$ are arbitrary reference values.

- "Quasilinearisation" (Philip, 1968) which assumes that the hydraulic conductivity function can be adequately described by a function of the form

$$K \propto e^{a\Psi} \qquad (5)$$

The exponent α is described by Philip (1968) as "a measure of the relative importance of gravity and capillarity for water movement", and is unique for a particular soil.

- Full linearisation (Pullan,1990), assuming the diffusivity function to have a single, constant value for all moisture contents.

Considering flow in two dimensions only, Richards equation may be thus be written in the form

$$D_x\frac{\partial^2\theta}{\partial x^2} + D_z\frac{\partial^2\theta}{\partial z^2} - \alpha\, D_z\frac{\partial\theta}{\partial z} = \frac{\partial\theta}{\partial t} \qquad (6)$$

The consequences of constrained diffusivity are significant and are noted in Fityus and Smith (1994). The advantage, however, is that the coefficients of the derivatives are now constant, and analytical solutions via the transform method can be found.

3. ANALYTICAL SOLUTION

Direct analytical solutions to the partial differential equation (6) are generally not possible. It can be solved numerically using finite element techniques.

Alternatively, mathematical transformations can be applied to reduce equation (6), with partial derivatives in the two spatial variables and the time variable, to an ordinary differential equation in two transformed variables with derivatives with respect to the depth variable only. These methods are similar to those applied by Small et. al.(1989), in the solution of the consolidation equation.

Partial derivatives with respect to time are eliminated by the Laplace transformation (7), where $\bar{\theta}$ denotes the transformed moisture variable.

$$\bar{\theta}(z,x,s) = \int_0^{\infty} e^{-st}.\theta(z,x,t)\, dt \qquad (7)$$

Application of the derivative theorem (Doetsch, 1970) associated with this transformation effectively replaces partial derivatives with respect to t, with functions of a new, complex variable, s.

In a similar way, partial derivatives with respect to the horizontal coordinate, x, are eliminated using the Fourier transform

$$\bar{\Theta}(z,\xi,s) = \int_{-\infty}^{\infty} e^{-i\xi x}.\bar{\theta}(z,x,s)\, dx \qquad (8)$$

and its inverse

$$\bar{\theta}(z,x,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}.\bar{\Theta}(z,\xi,s)\, d\xi \qquad (9)$$

where $\bar{\Theta}$ denotes the Fourier transformed moisture variable.

Application of the derivative theorem (Bracewell, 1986) associated with this transformation effectively replaces partial derivatives with respect to x, with functions of a new complex variable, ξ. For problems symmetric in x, ξ reduces to a real variable.

After successive transformations, equation (6) becomes

$$D_z\frac{d^2\bar{\Theta}}{dz^2} - a.D_z\frac{d\bar{\Theta}}{dz} - (\xi^2 D_x + s)\bar{\Theta} = 0 \qquad (10)$$

It is possible to find exact analytical solutions to equation (10) for a single layer, in terms of the transform variables. Vertically non homogeneous soil profiles can be handled by assembling single, homogeneous layer solutions into a finite layer formulation, where the boundary condition between layers maintains continuity of potential.

The transformed solutions generated must then be inverted to yield solutions at the required coordinates and required times. Because of the complicated analytical form of the solution expressions, inversions for each of the transformations are carried out numerically. Inversion of the Laplace Transform uses the method of Talbot (1979). Inversion of the Fourier transform is achieved either by the numerical evaluation of the inversion integral in equation (9), using Gaussian quadrature. (Small et.al., 1988) or using Fast Fourier Transform inversion routines. While the former method is has the advantage of being able to produce solutions at uniquely specified locations, the latter method has been found to be more convenient and efficient. This is despite having to generate a large number of solutions at regularly spaced points, which do not necessarily coincide with the locations at which they are required.

4. BOUNDARY CONDITIONS

Accurate modelling of field problems requires realistic simulation of boundary behaviour. The available numerical techniques are somewhat limited when compared to observed climatic phenomena. Two convenient and useful approximations involve the prescription of either constant fluxes or constant moistures at the bounding surfaces of the layer. These are sometimes referred to as Type 1 and Type 2 boundary conditions, respectively.

The applicability of the Type 1 and Type 2 boundary conditions is discussed in Fityus and Smith (1995). Also discussed is the inherent difficulty in the direct application of the Type 2 condition with the transform solution approach. The difficulty arises because the transform method of solution accommodates the application of only one type of boundary condition across a layer surface. The only valid condition beneath a cover is that of a zero flux, and this cannot be simultaneously applied with a constant moisture content prescribed for the uncovered areas. Fityus and Smith (1995) suggested that a possible approach to overcome this difficulty might involve the use of a Type 3 boundary condition (also described as "linear" by Carslaw & Jaeger, 1959). This is indeed the case and its application is discussed in section 5.

5. THE TYPE 3 BOUNDARY CONDITION

The Type 3 boundary condition, (11), is actually a flux boundary condition where the flux is continuously proportional to the difference between the instantaneous surface moisture content, $\vartheta_{Top}(x,t)$, and some prescribed limiting value, ϑ_a.

$$V(x,t) = h(x).\ (\vartheta_{Top}(x,t) - \vartheta_a) \qquad (11)$$

Expressed in terms of a moisture change, $\Delta\vartheta_{Top}$, where the change is the difference between the transient and initial values($\vartheta_{Top}(x,t)$ and ϑ_{To} respectively) we get

$$V(x,t) = h(x).\ (\Delta\vartheta_{Top}(x,t) + \vartheta_{To} - \vartheta_a) \qquad (12)$$

The function, h, may be prescribed to control both the magnitude of the resulting flux and the rate at which the limiting moisture value is approached. Special cases exist for particular h values: h equal to zero results in the zero flux condition required beneath covers; h very large causes instantaneous convergence to the limiting moisture and thus approximates a Type 2 condition. (Note that, in (11), h will always be negative or zero)

Successful application of the Type 3 boundary condition requires that it hold at all points across the surface of the layer, at all times. As the surface moistures vary in both position and time, a special solution formulation is required.

Taking the Laplace transform of (12) yields

$$\overline{V}(x,s) = \dot{h}(x).\ (\Delta\overline{\theta}_{Top}(x,s) + \overline{\theta}_{To} - \overline{\theta}_a) \qquad (13)$$

where $\Delta\overline{\theta}$ again denotes the Kirchhoff and Laplace transformed moisture variable and $\dot{h}(x) = \frac{h(x)}{D_z}$.
Fourier transformation of (13) would require the evaluation of convolution integrals, resulting from the product of two spatially varying functions. This can be avoided, however, using the superposition of partial solutions to yield an accurate approximation to the full solution. This is achieved by constraining the transient surface moisture change to a spatially constant value, and adopting a pulse function for $h(x)$. In this way, (13) describes a flux which is zero everywhere across the layer surface, except for a discrete "segment" where it is constant. A set of mutually exclusive segments are defined to cover the entire surface.

As $\Delta\overline{\theta}_{Top}$ is now constant, a Fourier Transformation can now be applied to (13) without producing a convolution. Solutions at each surface node can be calculated for each applied flux segment. These can then combined so as to simultaneously satisfy the Type 3 condition at a subset of nodes corresponding to the centre of each segment. This approach is illustrated in Figure 1. and described as follows:

- The surface boundary is divided up into a series of spatially discrete regions (segments), with the edge of the cover coinciding with the boundary of a segment.(Figure 1.c.) Let the number of segments be n; in this instance, n = 12.
- A flux of unit intensity is then applied to each of the segments ($i = 1$ to n), in turn.(Figure 1.d.) The application of each flux results in a change in moisture, $\Delta\overline{\theta}$, at the central node of all of the defined segments. Let the change beneath node j, due the flux applied over segment i, be $\Delta\overline{\theta}_{ji}$. At the central node, beneath the applied flux, this change, $\Delta\overline{\theta}_{ii}$, is large; beneath remote segments, $\Delta\overline{\theta}_{ji}$ is relatively small.
- The segments of applied flux must then scaled by multipliers A_i ($i = 1$ to n) so that the moisture changes at each central node, due to all of the unit flux segments, satisfy the Type 3 condition. The Type 3 condition, formulated for each of the segments, gives,

$$A_i = h(x).\left[\sum_{j=1}^{n}(A_j\Delta\overline{\theta}_{ij}) + \overline{\theta}_{To_i} - \overline{\theta}_a\right],\ i = 1, n \qquad (14)$$

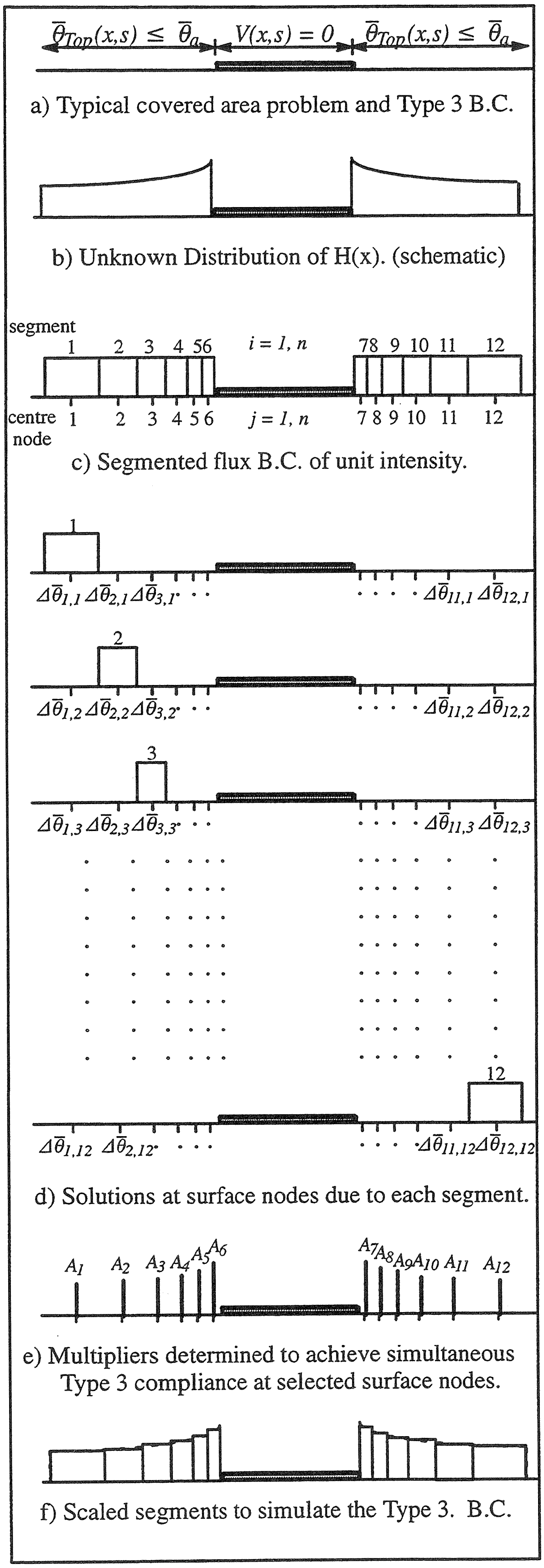

Figure 1. Schematic illustration of an approximated Type 3 Boundary Condition.

- The multipliers (Figure 1.e).) are given by the simultaneous solution of equations (14). The equations are rearranged in matrix form in (15).

$$\begin{bmatrix} \bar{\theta}_{To1}-\bar{\theta}_a \\ \bar{\theta}_{To2}-\bar{\theta}_a \\ \bar{\theta}_{To3}-\bar{\theta}_a \\ \vdots \\ \vdots \\ \bar{\theta}_{Ton}-\bar{\theta}_a \end{bmatrix} = \begin{bmatrix} \Delta\bar{\theta}_{11}-\frac{1}{h} & \cdots & \Delta\bar{\theta}_{n1} \\ \Delta\bar{\theta}_{12} & \cdots & \Delta\bar{\theta}_{n2} \\ \Delta\bar{\theta}_{13} & \cdots & \Delta\bar{\theta}_{n3} \\ \vdots & \ddots & \vdots \\ \Delta\bar{\theta}_{1n} & \cdots & \Delta\bar{\theta}_{nn}-\frac{1}{h} \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{n-1} \\ A_n \end{bmatrix} \quad (15)$$

- The solutions at all depths, determined for each unit flux segment, are then scaled by their appropriate multiplier (Figure 1.f).) and linearly combined to yield an approximate solution.
- Note that segments beneath the cover are omitted from the procedure as their scaling factors must be zero to achieve the necessary zero flux condition in this region.

6. FORMULATION OF THE LEAKING PIPE PROBLEM

The transform solution formulation method can be extended to accommodate the presence of a leaking pipe at any point in the soil layer. The pipe inflow may be modelled either as a Type 1 or a Type 3 phenomenon.

The Type 1 approach is easily implemented by applying a constant inflow over a small area, at the required depth. In the finite layer formulation, it appears as a non zero term at an appropriate node (between defined layers) in the prescribed flux vector. The value of the term is given by the Laplace transformation of a pulse function of finite width and appropriate intensity.

Special consideration must be given when a leaking pipe is simulated in a profile with a Type 3 condition specified at the surface. This is because the pipe is a moisture source which can produce changes in moisture at surface nodes, distinct from those due to the surface inflows. The changes produced at the surface by a Type 1 leaking pipe are independent of the changes produced by the Type 3 surface condition. They can therefore be considered separately. The formulation is thus extended to firstly evaluate the Type 1 changes and then satisfy the Type 3 condition at the surface to include these effects. This involves the addition of a moisture change term to each term in the left-hand vector in (15), equal to the change in moisture caused by the pipe.

The implementation of a pipe leaking according to a Type 3 condition is slightly more involved. In this case, the predicted change in moisture must consider the existing moisture content in the region of the pipe as well as changes which occur due to inflows at the surface and changes due to the pipe itself. For a Type 1 surface condition, the effects at the pipe can be evaluated independently. Once determined, the appropriate flux to satisfy the Type 3 requirements at the pipe can be calculated directly.

For Type 3 conditions at both the surface and the pipe, the full solution must be determined simultaneously. This can be achieved by treating the pipe in the same way as each of the segments at the surface. The net effect is to extend the vectors in (15) by one row and the matrix by an extra row and column. This yields

$$\begin{bmatrix} \bar{\theta}_{To1}-\bar{\theta}_a \\ \bar{\theta}_{To2}-\bar{\theta}_a \\ \bar{\theta}_{To3}-\bar{\theta}_a \\ \vdots \\ \bar{\theta}_{Ton}-\bar{\theta}_a \\ \bar{\theta}_{Po}-\bar{\theta}_{ap} \end{bmatrix} = \begin{bmatrix} \Delta\bar{\theta}_{11}-\frac{1}{h} & \cdots\cdots & \Delta\bar{\theta}_{1n} & \Delta\bar{\theta}_{1p} \\ \Delta\bar{\theta}_{21} & \cdots\cdots & \Delta\bar{\theta}_{2n} & \Delta\bar{\theta}_{2p} \\ \Delta\bar{\theta}_{31} & \cdots\cdots & \Delta\bar{\theta}_{3n} & \Delta\bar{\theta}_{3p} \\ \vdots & \ddots & \vdots & \vdots \\ \Delta\bar{\theta}_{n1} & \cdots\cdots & \Delta\bar{\theta}_{nn}-\frac{1}{h} & \Delta\bar{\theta}_{np} \\ \Delta\bar{\theta}_{p1} & \cdots\cdots & \Delta\bar{\theta}_{pn} & \Delta\bar{\theta}_{pp}-\frac{1}{h} \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{n-1} \\ A_n \\ A_p \end{bmatrix} \quad (16)$$

7. TYPICAL SOLUTIONS

Figure 2. shows a typical set of solutions for infiltration around a cover on a soil profile. The properties of the clay layer are those of the "Yolo light clay" described by Moore (1939). The soil layer is 1 m. deep and the cover is 6 m. wide. A leaking pipe is present at a depth of 0.5 m. and positioned 0.1 m. beyond the left-hand edge of the cover. The boundary conditions are prescribed as follows.

- The surface boundary condition is of Type 3 with the limiting moisture content, ϑ_a, equal to the saturated moisture content, 0.495. The value of h is large (–10000), effectively causing the surface moisture to reach saturation at the instant the event begins, thereby simulating a Type 2 condition.
- The base boundary condition is of Type 2 with zero change in moisture content prescribed.

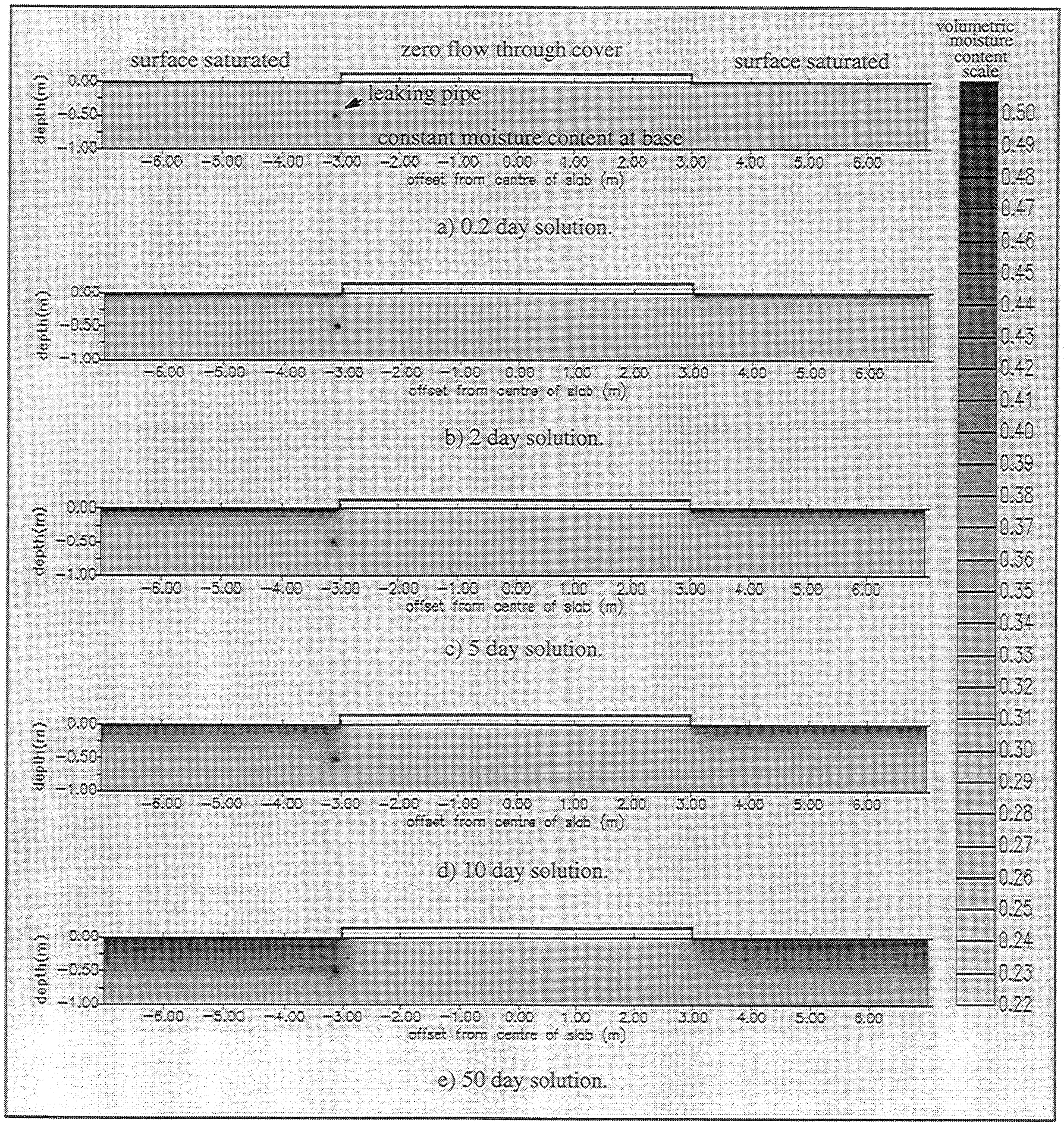

Figure 2. Typical solution of a covered area with a Type 3 boundary condition.

- The condition at the pipe is of similar type to that at the surface: a Type 3 with a large h to simulate instantaneously maintained saturation.

8. FUTURE DEVELOPMENTS

Work is already advanced on a modified formulation which will enable a sequence of different boundary events to be applied in succession. In this way, a climatic record consisting of any number of infiltration and evaporation events can be modelled. Further, the pipe can begin to leak at any intermediate time, and different combinations of surface and pipe boundary condition types can be selected, according to the nature of the event being modelled

9. DISCUSSION AND CONCLUSIONS

The transform solution approach to the unsaturated soil moisture problem provides a fast and efficient alternative to purely numerical methods. Spatial variations in boundary condition types are possible using the Type 3 condition, in which the flux is set proportional to the difference between the the instantaneous moisture and some limiting value. This can be applied as an approximation in which the condition is strictly satisfied at a subset of surface nodes by the linear superposition of boundary condition 'segments'.

The accuracy of the approximation is influenced by the number of segments into which the boundary is divided. Preliminary indications are that 16 segments, becoming narrower adjacent to the edge of the slab, are sufficient to produce accurate solutions. If use is made of symmetry in the formulation, this can be reduced to 8 pairs of segments.

The Type 3 formulation thus requires the evaluation of a number of discrete, full solutions for boundary conditions of restricted extent: one for each segment. In contrast, the Type 1 condition can be evaluated in a single step, with the full width of the region treated as a single segment. As a consequence, the Type 3 boundary condition is significantly less efficient than the Type 1. Despite this, both approaches are fast when compared with time stepping numerical formulations. The speed advantage increases with the length of the events being modelled This is because both Type 1 and Type 3 transform solutions require no iteration for a given set of boundary conditions, regardless of how long they are maintained.

Valid, quantitative comparisons of efficiencies cannot yet be made between transform and finite element solutions. The relative efficiency of the transform solution method cannot be accurately quantified until a more generalised solution is achieved. This will involve completion of the developments discussed in section 8.

10. ACKNOWLEDGMENTS

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