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Elastic-plastic Work-hardening Model for Soft Clays

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SUMMARY Constitutive equations for a simple cap model using classical plasticity theory are presented. Comparisons of stress-strain curves predicted by the finite element analysis with the experimental data are shown. A good agreement is obtained for the Grundite clay tested and modelled.

1 INTRODUCTION

For prediction of soil deformations, it is necessary to consider the stress-strain behavior of the soil. Linear elastic representation has been much used because of its simplicity and the fact that solutions to boundary value problems in linear elasticity were available. However, soil behavior is known to be path dependent and non-linear. Moreover, even at small loads, soils exhibit significant irrecoverable deformation. Thus, results from a linear elastic analysis may be misleading. In this paper, a simple elastic-plastic model, essentially a variant of DiMaggio and Sandler's (1971) formulation, is introduced. Characterization of the proposed model through simple laboratory tests is discussed, and the stress-strain relations explicitly developed. Finally, adequacy of the model is examined in application to tests on Grundite clay.

2 STRESS-STRAIN RELATIONS IN PLASTICITY

For small deformations, the total strain can be regarded as the sum of recoverable and irrecoverable parts (Felippa, 1966), i.e.:

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon''_{ij} \tag{1}$$

where ε_{ij} are components of the infinitesimal strain tensor, and superscripts ' and " denote, respectively, the elastic part and the plastic or irrecoverable part of the strain.

NOTE: Roman subscripts take on the range of values 1, 2, 3. Summation on repeated indices is implied. δ_{ij} is Kronecker's delta.

The incremental stress is related to the incremental elastic strain by the equation:

$$\dot{\sigma}_{ij} = E_{ijk\ell}\,\dot{\varepsilon}'_{k\ell} \tag{2}$$

where $E_{ijk\ell}$ is the fourth rank elasticity tensor. Writing $\dot{\varepsilon}'_{k\ell} = \dot{\varepsilon}_{k\ell} - \dot{\varepsilon}''_{k\ell}$ and using the normality rule:

$$\dot{\sigma}_{ij} = E_{ijk\ell}\,\dot{\varepsilon}_{k\ell} - E_{ijk\ell}\,\lambda\,\frac{\partial f}{\partial \sigma_{k\ell}} \tag{3}$$

where f is the yield function and λ is a positive scalar. For loading:

$f(\sigma_{ij}, \varepsilon''_{ij}) = 0$ which implies:

$$\frac{\partial f}{\partial \sigma_{ij}}\,\dot{\sigma}_{ij} + \frac{\partial f}{\partial \varepsilon''_{ij}}\,\dot{\varepsilon}''_{ij} = 0 \tag{4}$$

Using normality rule, Eq. (4) gives:

$$\frac{\partial f}{\partial \sigma_{ij}}\,\dot{\sigma}_{ij} + \frac{\partial f}{\partial \varepsilon_{ij}}\,\dot{\lambda}\,\frac{\partial f}{\partial \sigma_{ij}} = 0 \tag{5}$$

Substituting Eq. (3) in Eq. (5):

$$\frac{\partial f}{\partial \sigma_{ij}}\,E_{ijk\ell}\,\dot{\varepsilon}_{k\ell} = B\,\lambda \tag{6}$$

where

$$B = -\frac{\partial f}{\partial \varepsilon''_{ij}}\frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \sigma_{ij}}\,E_{ijk\ell}\,\frac{\partial f}{\partial \sigma_{k\ell}} \tag{7}$$

Hence,

$$\lambda = \frac{\partial f}{\partial \sigma_{ij}}\,\frac{E_{ijk\ell}\,\dot{\varepsilon}_{k\ell}}{B} \tag{8}$$

Substituting Eq. (8) in Eq. (3):

$$\sigma_{ij} = E_{ijk\ell}\left(\dot{\varepsilon}_{k\ell} - \lambda\,\frac{\partial f}{\partial \sigma_{k\ell}}\right)$$

$$= E_{ijk\ell}\,(\delta_{km}\,\delta_{\ell n} - L_{k\ell mn})\,\dot{\varepsilon}_{mn} \tag{9}$$

where

$$L_{\ell kmn} = \frac{\partial f}{\partial \sigma_{ij}}\,\frac{E_{ijk\ell}\,\dfrac{\partial f}{\partial \sigma_{mn}}}{B} \tag{10}$$

This formulation obtained by Felippa (1966) relates the total stress increment to the total strain increment and is valid for all cases including perfect plasticity.

3 PROPOSED MODEL FOR SOIL

The model proposed by the writers is a modification of DiMaggio and Sandler's (1971) model using some of the concepts from the earlier investigations at

M.I.T. and Cambridge. The objective is a model which is easy to characterize using data from standard tests. Figure 1 shows the model consisting of

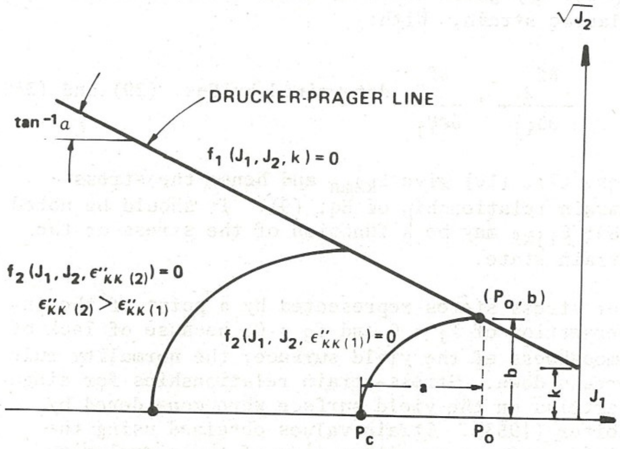

Figure 1 Proposed Model

the Drucker-Prager cone (the fixed yield surface):

$$f_1 = \alpha J_1 + J_2^{\frac{1}{2}} - k = o \tag{11}$$

and the hardening yield surface:

$$f_2 = b^2(J_1 - P_0)^2 + a^2J_2 - a^2b^2 = 0 \tag{12}$$

$f_2 = 0$ is a family of ellipsoids ordered by parameters P_0, a, b. This is identical to the assumption made by Christian (1966). In the J_1, $J_2^{\frac{1}{2}}$ plane, the hardening yield surface is a half-ellipse with center at $(P_0, 0)$. The parameters α, k completely define f_1 and exist along with the axis ratio:

$$R = \frac{a}{b} \tag{13}$$

also define the hardening yield function f_2. For a given state of the material defined by ε''_{kk}, the cumulative volumetric plastic strain, the intersection of the surfaces $f_1 = 0$, $f_2 = 0$ in the J_1, $J_2^{\frac{1}{2}}$ plane is (P_0, b). Then:

$$\alpha P_0 + b = k \tag{14}$$

i.e.:

$$b = k - \alpha P_0 \tag{15}$$

From Eqs. (13) and (15):

$$a = bR = (k - \alpha P_0) R \tag{16}$$

Substituting Eq. (15), (16) in Eq. (12):

$$(J_1 - P_0)^2 + R^2J_2 - (k - \alpha P_0)^2 R^2 = 0 \tag{17}$$

Rearranging terms, Eq. (17) gives:

$$AP^2_0 - BP_0 - C = 0 \tag{18}$$

where:

$$A = 1 - \alpha^2R^2 \tag{19}$$

$$B = 2(J_1 - R^2k\alpha) \tag{20}$$

$$C = R^2k^2 - R^2J_2 - J^2_1 \tag{21}$$

α, k, R completely define A, B. C. Solution to Eq. 18 is:

$$P_0 = \frac{B + \sqrt{B^2 + 4\ AC}}{2A} \quad \text{if } A \neq 0 \tag{22}$$

and

$$P_0 = -\frac{C}{B} \quad \text{if } A = 0 \tag{23}$$

For known P_0; a, b are defined by Eqs. (15), (16). Hence $f_2 = 0$ is completely defined by P_0. A relationship between P_0 and the volumetric plastic strain associated with f_2 will now be established.

The intersection of the hardening surface $f_2 = 0$ with the J_1 axis is say $(P_c, 0)$. Obviously

$$P_c = P_0 - a = P_0 - (k - \alpha P_0) R \tag{24}$$

$$= P_0(1 + \alpha R) - kR$$

P_c represents the maximum hydrostatic stress the material has been subjected to in its history. Considering Figure 2, let λ, κ respectively be compression index and the swelling index. If P_0, e_0 represent a reference state and p, e the 'current' states corresponding to $P_c = 3p$, the increment in plastic volumetric deformation is the intercept between the two lines at P_0. Thus:

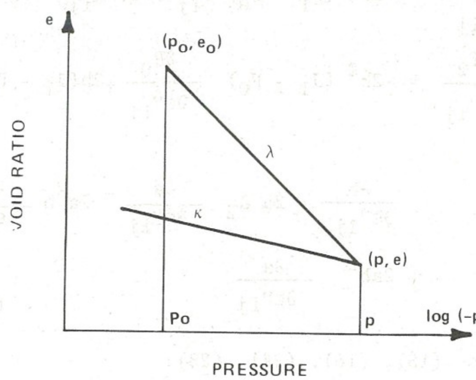

Figure 2 Idealized e - log p Curves for Isotropic Consolidation and Rebound

$$\varepsilon''_{kk} = -\frac{\lambda - \kappa}{1 + e_0} \log_{10} \frac{P}{P_0} \tag{25}$$

$$= - D\ (\ell_n P_c - \ell_n\ 3P_0) \tag{26}$$

$$= - D\ell_n\ P_c + B \tag{27}$$

Where ℓ_n stands for natural logarithm, $D = \frac{1}{2.3} \frac{\lambda \quad \kappa}{1 + e_0}$ and $B = D\ell_n\ 3P_0$ are coefficients which can be determined by experiments. Figure 3 shows a typical plot of Eq. (26). Eqs. (24) and (26) relate P_0 to the plastic volumetric strain increment from a reference state.
Explicitly:

$$P_c = \exp\left(\frac{B - \varepsilon''_{kk}}{D}\right) \tag{28}$$

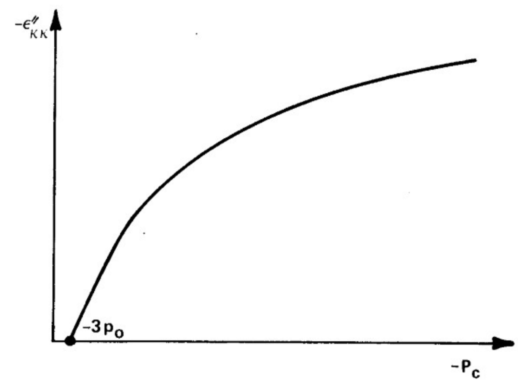

Figure 3 Plastic Volumetric Strain for Different Ellipses

4 STRESS-STRAIN RELATIONS FOR THE PROPOSED MODEL

Eq. (9) represents the general incremental stress-strain relationships. For the fixed yield surface $f_1 = 0$ defined by Eq. (11), the stress-strain relations have previously been developed, Baker et al., (1969); Reyes (1966), and will not be reproduced here. For the hardening cap defined by Eq. (12):

$$\frac{\partial f_2}{\partial \sigma_{ij}} = 2b^2 (J_1 - P_0)\, \delta_{ij} + a^2 s_{ij} \tag{29}$$

$$\frac{\partial f_2}{\partial \varepsilon''_{ij}} = -2b^2 (J_1 - P_0) \frac{\partial P_0}{\partial \varepsilon''_{ij}} + 2b(J_1 - P_0)^2 \frac{\partial b}{\partial \varepsilon''_{ij}} + 2a\, J_2 \frac{\partial a}{\partial \varepsilon''_{ij}} - 2a^2 b \frac{\partial b}{\partial \varepsilon''_{ij}} - 2ab^2 \frac{\partial a}{\partial \varepsilon''_{ij}} \tag{30}$$

From Eqs. (15), (16), (24), (28):

$$\frac{\partial P_0}{\partial \varepsilon''_{ij}} = \frac{\partial P_c}{(1 + \alpha R)\partial \varepsilon''_{ij}} = - \frac{P_c}{D(1 + \alpha R)} \delta_{ij} \tag{31}$$

$$\frac{\partial b}{\partial \varepsilon''_{ij}} = \frac{1}{R} \frac{\partial a}{\partial \varepsilon''_{ij}} \tag{32}$$

$$\frac{\partial a}{\partial \varepsilon''_{ij}} = R \frac{\partial b}{\partial \varepsilon''_{ij}} = \frac{\alpha R P_c}{D(1 + \alpha R)} \delta_{ij} \tag{33}$$

Substituting Eq. (31), (32), (33) in Eq. (30):

$$\frac{\partial f_2}{\partial \varepsilon''_{ij}} = \frac{2bP_c}{D(1 + \alpha R)} \delta_{ij}]\, (J_1 - P_0) \{b + \alpha(J_1 - P_0)\} + \alpha R^2 (J_2 - 2b^2)] \tag{34}$$

For a given stress state defined by J_1, J_2, material constants α, κ, D, B, and assumed ratio R, Eqs. (15), (16), (24), (28) completely define a, b, P_0, P_c for any given value of cumulative volumetric plastic strain. With:

$$\frac{\partial f_2}{\partial \sigma_{ij}}, \quad \frac{\partial f_2}{\partial \varepsilon''_{ij}} \quad \text{determined by Eqs. (29) and (34)}$$

Eqs. (7), (10) give $L_{k\ell mn}$ and hence the stress-strain relationship of Eq. (9). It should be noted that $E_{ijk\ell}$ may be a function of the stress or the strain state.

For stress states represented by a point of the intersection of $f_1 = 0$ and $f_2 = 0$, because of lack of smoothness of the yield surface, the normality rule breaks down. Stress-strain relationships for singularities on the yield surface were considered by Koiter (1953). Strain values obtained using the yield surfaces on either side of the singularity point are added. Figure 4 depicts the situation.

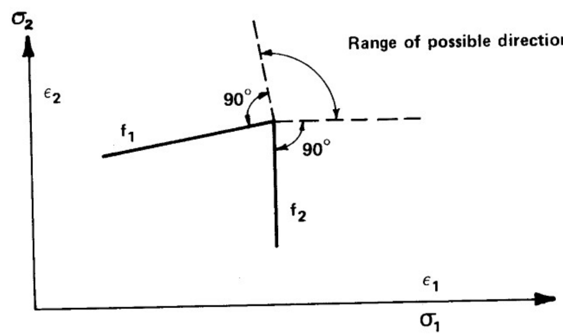

Figure 4 Corners in a Yield Surface, Graphical Interpretation of Koiter's Generalization

5 VERIFICATION OF THE PROPOSED MODEL

The elastic-plastic model proposed was implemented in a finite element computer program, Singh et al., (1976). This program was used to generate stress-strain curves for triaxial tests on the Grundite clay. The samples were consolidated isotropically to 2.5 Kg cm^{-2} and then loaded in the following stress paths:

IDC - Isotropically Consolidated Drained Compression Test

IDE - Isotropically Consolidated Drained Extension Test

IOC - Isotropically Consolidated Octahedral Stress Constant Test

Table I shows the material constants for the Grundite clay investigated under this study. These were obtained from an isotropic hydrostatic consolidation test and drained compression tests on isotropically consolidated specimens. The axis ratio R was assumed to be 3.5.

TABLE I

MATERIAL CONSTANTS FOR GRUNDITE CLAY

Property	Symbol	Value for Grundite Clay
Young's Modulus	E	60 Kg cm^{-2}
Poisson's Ratio	ν	0.40
Compression Index	λ	0.39
Rebound Index	κ	0.15
Cohesion	c	0.0
Angle of Internal Friction	ϕ	25.0
Constant	D	0.0505
Axis Ratio	R	3.5

Figure 5 shows the results of isotropic consolidation tests for Grundite clay. Figures 6 through 8

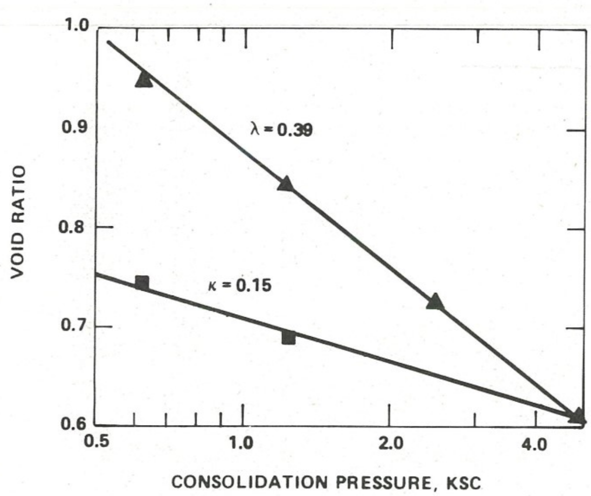

Figure 5 Isotropic Consolidation Test: Grundite Clay

show comparison of the stress-strain data predicted by the finite element analysis for various stress paths with the experimental data.

6 CONCLUSIONS

A simplification of DiMaggio and Sandler's model for elastic-plastic behavior of soils is proposed. The chief merit of the proposed model is that while conforming to the postulates of theory of plasticity, it requires only routine tests to completely characterize a soil. An assumption is required for the value of the axis ratio R. The performance of the model is seen to be quite satisfactory.

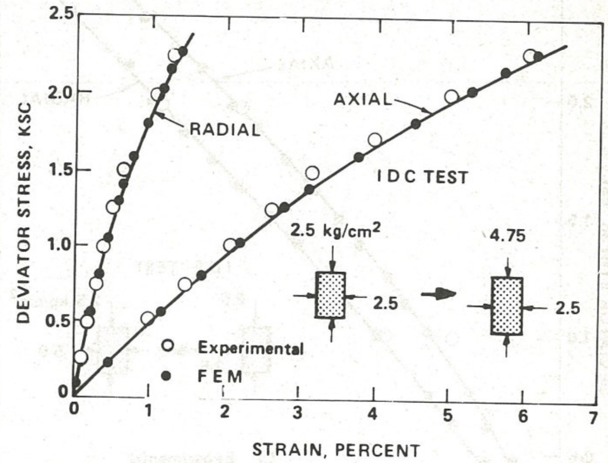

Figure 6 Stress-Strain Curve for Grundite Clay in IDC Test

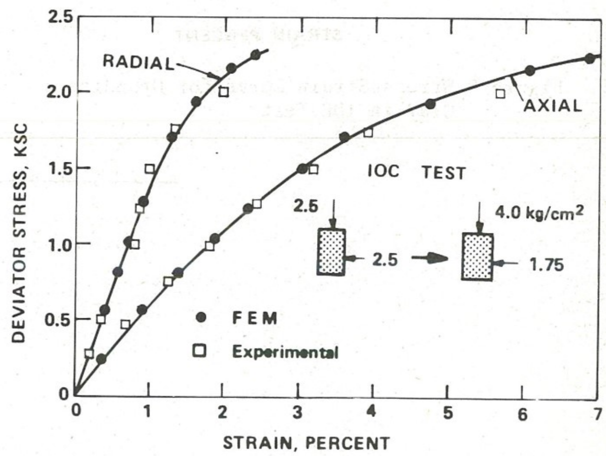

Figure 7 Stress-Strain Curve for Grundite Clay in IOC Test

7 REFERENCES

BAKER, L.E., SANDHU, R.S., and SHIEH, W.Y. (1969). Application of Elasto Plastic Analysis in Rock Mechanics by Finite Element Method. Proceedings, Eleventh Symposium on Rock Mechanics, Berkeley, California.

CHRISTIAN, J.T. (1966). Plane Strain Deformation Analysis of Soil. Contract No. 3-129, Report 3, Contract DA 22-079-eng-471, Department of Civil Engineering, M.I.T.

DIMAGGIO, F.L. and SANDLER, I.S. (1971). Material Model for Granular Soils. Journal of Engineering Mechanics Division, ASCE, Vol. 97, No. EM3, Proc. Paper 8212, June 1971, pp. 935-950.

FELIPPA, C. (1966). Refined Finite Element Analysis of Linear and Nonlinear Two-Dimensional Structures. Ph.D. Thesis, University of California, Berkeley.

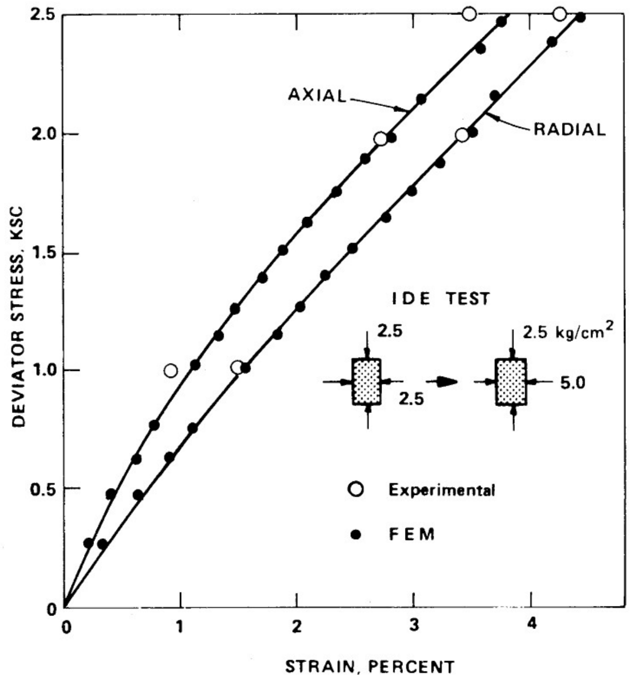

Figure 8 Stress-Strain Curve for Grundite Clay in IDE Test

KOITER, W.T. (1953). Stress-Strain Relations, Uniqueness and Variational Theorems for Elastic-Plastic Materials with a Singular Yield Surface. Quarterly of Applied Mathematics, Vol. 11, 1953, pp. 350-354.

REYES, S.F. (1966). Elastic-Plastic Analysis of Underground Openings by the Finite Element Method. Ph.D. Thesis, University of Illinois, Urbana.

SINGH, K.J., SINGH, R.D. and SANDHU, R.S. (1976). A Computer Program for Finite Element Analysis of Elastic-Plastic Work-Hardening Materials. Geotechnical Engineering Report No. 3, Department of Civil Engineering, The Ohio State University, Columbus, Ohio.