A Technique for the Back Analysis of Slope Failures

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SUMMARY A technique for the back analysis of slope failures is presented. The technique determines the geomechanical parameters such as cohesion and angle of friction of soil layers in a failed slope at limiting equilibrium by searching for an optimum in an N-dimensional space defined by the N unknown geomechanical parameters. The convergence to the optimum is achieved when the factor of safety determined by the relevant limit equilibrium method is equal to 1.00. Illustrative examples are given and modifications required on existing limit equilibrium computer programs to incorporate the technique are explained.

1 INTRODUCTION

In slope stability analysis, the factor of safety is defined as the ratio of the mobilised shear resistance over the driving force tending to produce failure. Its calculation can be based on a wide range of methods, such as Bishop's modified method, Morgenstern and Price's, the two-wedge method, etc. depending on the mode of slope failure and the user's option. It immediately follows from this definition that the slope is safe when the factor of safety is greater than 1.0, and slope failure occurs when the driving force on the slope mass approaches a state of limiting equilibrium with the available shear resistance, or when the factor of safety is reduced to unity.

Back analysis of slope failures essentially involves the determination of various geomechanical parameters of soil layers in a failed slope at the instant of limiting equilibrium. The unknown geomechanical parameters of the soil layers may be one or any combination of: cohesion, angle of friction, pore pressure coefficient or the set of coordinates defining the piezometric surface, and the soil density. The failure surface was assumed or known to have taken place through these soil layers.

Back analysis is normally carried out to determine or to verify the geomechanical parameters for further design or remedial works. The parameters were then either not available or suspected to be erroneous. Important as it is in geotechnical engineering, back analysis of slope failures has surprisingly received little treatment, and is by-and-large carried out on a trial-and-error basis.

The following will present a technique, already well-known in the field of operations research, for the systematic evaluation of the geomechanical parameters required in a back analysis. The technique interprets the back analysis of slope failures as an optimisation problem with an optimum related to the factor of safety being equal to 1.00. The technique has recently been adapted with success in the search algorithms for the critical slope failure surface (Nguyen, 1983).

2 BACK ANALYSIS OF SLOPE FAILURES AND THE SIMPLEX REFLECTION TECHNIQUE

If the factor of safety \( F \) of a slope is designated as a function of a set of variables comprising the geomechanical parameters such as cohesion \( c \), angle of friction \( \phi \), and soil density \( \gamma \), then back analysis of slope failures can be mathematically interpreted as finding a solution of:

\[
F = F(c, \phi, \gamma, \ldots) = 1.00
\]

where \( c, \phi, \gamma, \) etc. are unknowns.

If we define an optimal function \( f \) as:

\[
f = | F - 1.00 | \geq 0
\]

then finding the minimum of function \( f \) will achieve identical results as the solution of equation (1) or a back analysis setting the safety factor function \( F \) equal to 1.00.

This equivalence is valid since the optimal function \( f \), as the absolute value of the difference between the safety factor and 1.00, is always positive and has an optimum (minimum) value of zero when the safety factor function \( F \) approaches unity from either side. Back analysis of slope failures can therefore be interpreted as an optimisation process and the search for an optimum of function \( f \) will also yield the values of parameters required by back analysis.

The simplex reflection technique described as followed is not the simplex method of linear programming. It was first coined by the authors (Spedley, Hext and Hinsworth, 1962) to describe a conceptual structure formed by a set of grid points in N-dimensional space. The structure is called the simplex and the grid points are the vertices of the simplex.

In back analysis of slope failures, we seek the minimum of function \( f(X) \)

\[
f_{\text{min}} = \min_{i} \{ f(X)_{i} \}
\]

where

\[
f(X)_{i} = | F(X)_{i} - 1.00 | , \text{ optimal function with } (X)_{i} \text{ as the } i\text{-th set of } N \text{ unknown geomechanical variables:}
\]

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\( (X)_\parallel = (x_1, x_2, x_3, \ldots, x_N)_\parallel \), a vector of \( N \) coordinates defining the \( \parallel \)-th vertex in an \( N \)-dimensional space. \( x_1, x_2, x_3, \ldots \) assume the physical meanings of unknown geomechanical parameters in the back analysis.

\( \text{e.g. } x_1 \text{: cohesion of soil layer} "M" \\
x_2 \text{: density of soil layer} "M" \\
x_3 \text{: angle of friction of soil layer} "M" \\
\)

Let's take the sake of simplicity consider a back analysis problem involving the determination of the cohesion, angle of friction, and density of the surface soil layer of a failed slope. The three-dimensional space for the simplex thus comprises a coordinate system of three axes, being

\[ x_1 : \text{cohesion of the surface soil layer}, c \\
x_2 : \text{angle of friction of the surface soil layer}, \phi, \text{ and} \\
x_3 : \text{density of the surface soil layer}, \gamma. \]

The first step in the simplex reflection method is to set up an initial grid or simplex which in this case is a regular tetrahedron with four equal sides and four vertices. (For a two-dimensional problem, the simplex is an equilateral triangle, and for an \( N \)-dimensional space, the simplex is a conceptual structure with \( (N+1) \) vertices, equidistant to each other).

The optimal function \( f \) is then evaluated at each of the four vertices in turn, by using the coordinate set defining the relevant vertex in the \( \psi_1-\psi_2-\psi_3 \) (i.e. \( x_1-x_2-x_3 \)) coordinate system. The values of function \( f \) at the four vertices (say \( f_1, f_2, f_3, \) and \( f_4 \)) are then compared. If \( f_1 \) is the largest or the worst value among the four, i.e.

\( f_1 > f_2 > f_3 > f_4 \)

then it can be reasoned that the minimum of function \( f \) would most likely be found in the region on the other side of a plane defined by the remainder three vertices 2, 3 and 4. The worst vertex (vertex 1, associated with \( f_1 \)) is then reflected across the centroid of the triangle formed by the remainder vertices (hereafter, called the basal centroid) to a new position, say point 5, and a new tetrahedron simplex, with vertices 2, 3, 4 and 5 is created (Figure 1). The value of function \( f \) at the new vertex 5 is evaluated. The values of function \( f \) at the four vertices of the new simplex (\( f_2, f_3, f_4 \) and \( f_5 \)) are next compared with each other, and the search for the direction towards the optimum is repeated, by the reflection of the vertex with the largest \( f \) value, until the optimum is found. In this application, the optimum is a minimum equal to zero.

In general, basic analysis by simplex reflection technique for \( N \) unknown parameters involves the following steps. (The algebra for these steps are presented in the Appendix)

**STEP 1:** Set up an initial simplex with \( (N+1) \) vertices equidistant to each other. Each vertex \( \parallel \) of the simplex is a point defined by

\( (X)_\parallel = (x_1, x_2, x_3, \ldots, x_N)_\parallel \)

where \( (X)_\parallel \) is the vector or coordinate set comprising the geomechanical variables.

**STEP 2:** Evaluate optimal functions \( f = |F-1.00| \), where \( F \) is the factor of safety, at each of the \((N+1)\) vertices in turn by using the relevant vector \( (X)_\parallel \) at each vertex.

Rearrange the \((N+1)\) values of function \( f \) evaluated at the vertices in descending order:

\[ f_1 > f_2 > f_3 > \ldots > f_{N+1} \]

Then reflect the vertex with the highest \( f \) value (i.e. Vertex 1) to the opposite side, across the basal centroid.

**STEP 3:** Check the optimisation constraints:

(i) If the reflection in **STEP 2** results in a new vertex with function \( f \) still the largest in the new simplex, further reflection will return the exact position of the old vertex, and the reflection algorithm is locked in a vacillation loop. To avoid this, the vertex with the second largest value in \( f \) is then reflected.

(ii) Alternatively, if the reflection results in a new vertex with infeasible coordinates, for example, negative values of cohesion, angle of friction, or soil density below a realistic limit (\( 10 \) kN/m\(^2\)), it is then not possible to evaluate function \( f \) at this vertex. There are numerous ways to overcome this type of constraint, and the adopted strategy is to reflect the infeasible vertex back to its previous region, but only a point located half-way between the basal centroid and the old vertex (Figure 2). The new simplex is no longer a regular structure.

(iii) If a single vertex remains stationary at one point (i.e. not reflected at all) after many successive reflections (e.g. Vertex 7 in Figure 3), simplex reflections merely circle around the stationary vertex. In this instance, it is a good probability that the search is nearing the optimum but the simplex size is too large for convergence. To overcome this, the simplex is shrunk in size with the best vertex retained, and the reflection algorithm resumed.

**STEP 4:** Repeat the simplex reflections in the manner described in **STEP 2** and **STEP 3** above until convergence.

(For further reading on the algorithm, see Burley, 1974: Beightler et al., 1979).

3 BACK ANALYSIS COMPUTING AND EXAMPLES

It has been revealed from the present study that the large part of computation time in back analysis is spent on the evaluation of the factor of safety (or the optimal function) at the vertices. The simple algebraic operations required by the simplex reflection technique thus render the technique adaptable to any computational facilities available, including pocket calculators.

For existing computer programs to determine the factor of safety based on limit equilibrium methods, only few modifications are required to incorporate the simplex reflection algorithm for back analysis:

(i) the part of the program used in computing the factor of safety should be separated from the part used for data input, and made into a subroutine.

(ii) the subroutine to calculate the factor of saf-
Figure 1 Simplex reflection for a 3-dimensional space.

Figure 2 Reflection from infeasible region.

Figure 3 Simplex shrinking (7-11-6) following the circling around a single vertex (No.7).

Figure 4 Embankment cross-section and reflection strategy for example 1.

Figure 5 Back analysis of spoil pile failures (after Richards et al, 1981).

(iii) The main program will have a part or another subroutine that will carry out the simplex reflection algorithm. This simplex subroutine will make CALLS to the factor-of-safety subroutine every time the determination of optimal function at a new vertex is required.

(iv) The first simplex is always a regular structure with vertices equidistant to each other. The alge-
bra used to set up this initial simplex requires that the units of geomechanical parameters (i.e., vertex coordinates) be of comparable order of magnitude. For example, if distance is in metres, angle should be in degrees, density in kN/m$^3$, cohesion in kPa, and pore pressure coefficient $r_u$ should be scaled up by a factor of 10.

(v) For a back analysis problem that requires the determination of a piezometric surface, the vertex coordinates then become actual vertical coordinates of a number of points defining the piezometric surface. The horizontal coordinates of these points are fixed while the vertical coordinates are variable. In this case, the search for the critical piezometric surface is similar to the search for the critical failure surface, also performed by the simplex reflection technique (Nguyen, 1983).

The following examples will serve to illustrate the simplex reflection algorithm:

- **EXAMPLE I**: Back analysis of a circular slip failure is required to determine the cohesion ($c_1$) and angle of friction ($\phi_1$) of an embankment on a silty soil layer (Figure 4). The geomechanical properties of the silty soil layer are known, and the density of the embankment is 18.0 kN/m$^3$. If trial values of 5 kPa and 30° are used for $c_1$ and $\phi_1$, respectively to set up an initial simplex with side length equal to 2.0, only 4 simplex reflections are required to achieve a vertex having optimal function $f$ equal to 0.020, corresponding to a factor of safety of either 0.900 or 1.020. If convergence is defined as achieving an accuracy on $f$ being equal to 0.02 then the vertex coordinates required in optimizing function $f$ are 3.9 kPa for cohesion and 26° for angle of friction.

These vertex coordinates are also the strength parameters in limiting equilibrium with the driving force on the slope mass at failure, hence the shear strength required in the back analysis.

In back analysing the same slip circle but instead of choosing 5 kPa and 30° as the initial trial parameters for setting up the simplex, we begin the reflection algorithm with other feasible pairs of $c_1$ and $\phi_1$, for example 8 kPa and 8°, the final optimal solutions for $c_1$ and $\phi_1$ may be totally differ-

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<td>$\phi_1$ deg</td>
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**TABLE 1**

**DEPENDENCE OF OPTIMAL SOLUTIONS ON THE INITIAL SIMPLEX COORDINATES**

**TABLE 2**

**BACK ANALYSIS USING TWO-WEDGE FAILURE DATA FROM RICHARDS ET AL (1981). (RICHARDS ET AL SOLUTION: $c_1 = 30$ 30 kPa, $\phi_1 = 3.46^\circ$, $c_2 = 100$ 110 kPa, and $\phi_2 = 36.38^\circ$)**

<table>
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<td>$\phi_2$ deg</td>
<td>35</td>
<td>36.3</td>
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</tr>
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ent from 3.9 kPa and 26° as earlier obtained (see Table 1). Thus there are many different combinations of strength or geomechanical parameters that can satisfy the condition of failure as defined by limit equilibrium methods. This feature is in fact well known by geotechnical engineers and confirmed here by the simplex reflection technique.

EXAMPLE II: This back analysis example illustrates that the simplex technique can be used in conjunction with any limit equilibrium methods to calculate the slope factor of safety. The example was drawn from actual back analyses carried out by Richards et al. (1981) on spoil pile failures in a strip coal mine. The mode of slope failure in this instance is that of a two-wedge type (Figure 5), already described elsewhere (Nguyen, 1984).

The aim of the back analysis is to determine cohesion and angle of friction of the basal failure plane ($c_1$, $\phi_1$) and the scarp failure plane ($c_2$, $\phi_2$) for the spoil pile with density of 18.5 kN/m$^3$ and geometrical configuration as shown in Figure 5. The simplex in the four-dimensional space defined by $c_1$, $\phi_1$, $c_2$ and $\phi_2$ is a conceptual structure of 5 vertices. The back analysis results were summarised and shown in part in Table 2. The first solution was in good agreement with Richards et al.’s results. The second solution serves to illustrate the multiple solutions of the optimal function $f$ and the use of simplex translation. The reflections first converged to a false optimum with value of $f$ still much higher than its known optimum of zero. The whole simplex was then shifted to a new feasible region either by halving one set of vertex coordinates or by increasing the simplex side length (i.e. vertex-to-vertex distance), and then resetting the whole simplex accordingly for further reflections.

DISCUSSION

It has been shown in the above examples, and consistently proven by numerous tests in this study, that back analysis by the simplex reflection technique only requires a few iterations for convergence. The convergence criterion is normally established as the condition when the entire simplex lies within an optimum region of the optimal function (Parkinson and Hutchinson, 1972). In back analysis, the minimum of the optimum function is always known a priori ($f_{min} = 0.0$) and since the accuracy required on the factor of safety, and hence on the optimum function, at failure is not so stringent as compared with that demanded by other optimisation problems such as the search for the critical failure surface, the convergence criterion can be much simplified. This partially explains the small number of iterations or simplex reflections required to achieve convergence in back analysis.

Back analysis by optimisation also confirmed a feature quite well known to geotechnical engineers. That is the factor of safety ($F$) can be equated to unity by numerous different combinations of geomechanical parameters substituted in the equation solving for $F$. In operations research parlance, optimal function $f$ is a multimodal function of geomechanical parameters and the convergence to an optimum depends to a large extent on the initial trial values of these variables. The multimodal feature of back analysis optimisation indicates that the reliability of geotechnical engineering judgement on back analysis results would be enhanced with the number of required parameters being reduced. In fact this is the state of current practice in back analysis: the soil density determination (or estimation) and the identification of the soil cohesiveness are carried out first in order to reduce the number of unknowns and to provide good trial values of cohesion and angle of friction.

The multimodal characteristic of the back analysis optimal function is thus an inherent property of slope failure back analysis and the simplex reflection technique always results in true minima. 5 CONCLUSIONS

Back analysis of slope failures can be interpreted as an optimisation process with an optimum known a priori. The simplex reflection technique adapted from optimisation studies proved to be a systematic and efficient algorithm for the search of geomechanical parameters in back analysis. The technique does not require elaborate computational procedures during iterations and can be performed by any computational facilities available, including pocket calculators. It also confirmed an old geotechnical engineering maxim that back analysis can provide many different solutions to the slope failure equation with factors of safety being equal to 1.0. In this aspect, the simplex reflection technique not only emphasises the role of geotechnical engineering judgement required in back analysis of slope failures but also presents geotechnical engineers with a wide range of solutions to exercise their engineering judgement.

6 REFERENCES


APPENDIX

ALGEBRA FOR SIMPLEX REFLECTIONS

The following formulæ were abstracted from the references on optimisation studies cited above, and slightly modified based on this study:

(i) Coordinates of simplex vertices.

For an N-dimensional space, if $\ell$ is chosen as the
length of a simplex side connecting the \( r \)th vertex and the \( s \)th vertex:

\[
\ell = \left( (x_1^r - x_1^s)^2 + (x_2^r - x_2^s)^2 + \ldots + (x_N^r - x_N^s)^2 \right)^{1/2}
\]

(superscripts \( r \) and \( s \) denote the two interconnected vertices and subscripts 1 to \( N \), the coordinate axes), then the coordinate vectors of vertices 1 to \( N+1 \) of the initial regular simplex are determined as:

\[
\begin{align*}
\vec{x}_1 &= (x_1, x_2, x_3, \ldots, x_N) \\
\vec{x}_2 &= (x_1 + p_N \ell, x_2 + q_N \ell, x_3 + p_N \ell, \ldots, x_N + q_N \ell) \\
\vec{x}_3 &= (x_1 + q_N \ell, x_2 + q_N \ell, x_3 + q_N \ell, \ldots, x_N + q_N \ell) \\
\vec{x}_{N+1} &= (x_1 + q_N \ell, x_2 + p_N \ell, \ldots, x_N + p_N \ell)
\end{align*}
\]

where

\[
\begin{align*}
p_N &= \frac{\sqrt{N+1} - 1 + N}{N \sqrt{2}} \\
q_N &= \frac{\sqrt{N+1} - 1}{N \sqrt{2}}
\end{align*}
\]

and recommended side length \( \ell = N + 1 \)

(ii) Simplex reflection

If vertex \( \vec{x}_m \) is reflected to a new position \( \vec{x}_n \), then \( \vec{x}_n \) can be calculated as

\[
\vec{x}_n = \vec{x}_m + \rho (\vec{x}_n - \vec{x}_m)
\]

where \( \rho = 2.0 \) for symmetrical (mirror) reflection

\( \rho = 1.5 \) for reflection from infeasible region (Fig.2)

and \( \vec{x}_n = \frac{1}{N} (\vec{x}_1 + \vec{x}_2 + \ldots + \vec{x}_N) = \text{basal centroid of the old simplex containing } \vec{x}_m \).

(iii) Shrinking the simplex

If \( \vec{x}_o \) is the best vertex retained from a simplex to be shrunk in size the shrunk vertices \( \vec{x}_s \) can be calculated as:

\[
\vec{x}_s = 0.5 (\vec{x}_r + \vec{x}_o) \text{ where } s, o = 1 \rightarrow N
\]

and \( \vec{x}_s = \text{vertices of old simplex.} \)

The study revealed that the simplex should be shrunk if one single vertex remains stationary after 2\((N+1)\) reflections.