Sensitivity Analysis of Laterally Loaded Piles

B.B. BUDKOWSKA
M.S.C.E., Ph.D.
Assoc. Professor, University of Windsor
C. SZYMCZAK
M.S.C.E., Ph.D.
Professor, University of Windsor (Visiting) and Technical University of Gdansk

SUMMARY The first order variations of an arbitrary displacement and an internal force at a specified cross–section due to some variations of the design variables are derived by means of the adjoint method. The pile cross–section dimensions, the pile material constants and the soil constants are considered to be the design variables. The sensitivity analysis presented is valid for both linear and nonlinear behaviour of the soil and pile material. In some numerical examples dealing with the linear structures the distributions of the underintegral coefficients of the design variable variations determining the increments of the lateral displacement, the angle of cross–section rotation at the pile top due to the changes of the design variables are given. The accuracy of the approximation of the increments of the quantities under consideration due to the design variable variations is also investigated.

1. NOTATION

A – area of the pile cross–section;
D_t – tangent flexural stiffness of the pile;
E – Young's modulus of elasticity of the pile material;
E_t – tangent modulus of elasticity of the pile material;
EJ – flexural stiffness of the pile;
F_v, F_ψ, K_v, K_ψ – underintegral coefficients;
k – stiffness of linear Winkler–type foundation;
k_t – tangent stiffness of the foundation;
L – pile length;
M – moment;
M_x – bending moment;
r – foundation reaction per unit length;
$\bar{s}$ – design variables vector;
v – lateral displacement of the pile;
y, z – coordinate axes;
δ – variational operator;
ϵ – longitudinal strain;
ψ – cross–section rotation;
σ – normal stress;
∂ – first partial derivative.

Subscripts:

a – imposed on adjoint beam;
o – at cross–section $z = z_0$;
, – first partial derivative.

Superscripts:

M – corresponding to moment M;
P – corresponding to concentrated load P;
′ – differentiation with respect to z.

2. INTRODUCTION

Internal forces and displacements in laterally loaded piles can be calculated using a simple one–dimensional idealization in conjunction with the beam–on–elastic–foundation approach (Das, 1990; and Desai and Kuppusamy, 1980). The three displacements of the pile are usually assumed to be uncoupled, thereby superimposition of their effects is possible. However, the internal forces and the displacements may vary due to arbitrary variations of the design variables such as the pile cross–section dimensions, the pile material constants and the soil constants, as well. To facilitate calculations of these changes, without the additional redesign, the first order variations of any displacement or any internal force at the specified cross–section of the pile due to arbitrary variation of the design variables are derived. The considerations are valid for both linear and nonlinear behaviour of the soil and pile material. The adjoint method developed by Mroz et al. (3) is adopted to this case.

The accuracy of the approximations of the pile displacements changes due to the variations of the flexural stiffness of the pile and the foundation stiffness constant by means of their first variations are also discussed. The linear beam resting on the Winkler–type elastic foundation is assumed in this case.

3. FIRST ORDER VARIATION OF DISPLACEMENTS AND INTERNAL FORCES

Consider a pile made of nonlinear elastic material and subjected to lateral loads as shown in Fig. 1.
To derive the first order variation of any displacement or any internal force at the specified cross–section of the pile, a simple idealization of one–dimensional beam made of nonlinear elastic material and resting on a nonlinear elastic foundation is used. Using the Bernoulli hypothesis, the increments of the strain $\delta\epsilon$ of the beam may be written as

$$\partial\epsilon = -y \ \delta v'' \tag{1}$$

Hence, the bending moment–curvature incremental relationship can be obtained

$$\delta M_x = -D_t \ \delta v'' \tag{2}$$

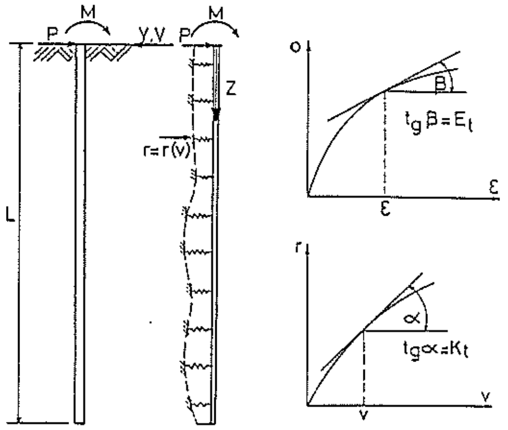

Figure 1. The laterally loaded pile and its model assumed; stress–strain relation $(\sigma-\epsilon)$ of the pile material; reaction–displacement relation (r–v) of the foundation

where

$$D_t = \int_A E_t\, y^2\, dA$$
$$E_t = \frac{d\sigma}{d\epsilon} \qquad (3)$$

Regarding the foundation, similarly the foundation reaction–displacement relationship can be written

$$\delta r = k_t\, \delta v \qquad (4)$$

According to the adjoint structure concept (Mroz et al., 1985; Szefer et al., 1987) an adjoint beam made of linear elastic material with the tangent modulus of E_t and resting upon the linear Winkler–type foundation described by the tangent foundation stiffness k_t corresponding to the actual stress and displacement fields of the pile and the foundation is introduced. The adjoint beam is subjected to a concentrated load P_a and a concentrated moment M_a at the cross–section $z = z_o$ in which the displacements variations are sought. Consider some statically admissible variations of the displacement field of the primary beam.
Application of the virtual work theorem (Washizu, 1974) yields

$$P_a\, \delta v_o + M_a\, \delta\, \psi_o = -\int_o^L M_{xa}\, \delta v''\, dz + \int_o^L r_a\, \delta v\, dz \qquad (5)$$

Taking into account that the statical fields depend on the actual displacements and the vector of the design variables variation $\delta\bar{s}$, the following linear relations are derived

$$\delta M_x = \frac{\partial M_x}{\partial v''}\,\delta v'' + \frac{\partial M_x}{\partial \bar{s}}\,\delta\bar{s} = D_t\,\delta v'' + M_{x,\bar{s}}\,\delta\bar{s}$$
$$\delta r = \frac{\partial r}{\partial v}\,\delta v + \frac{\partial r}{\partial \bar{s}}\,\delta\bar{s} = k_t\,\delta v + r_{,\bar{s}}\,\delta\bar{s} \qquad (6)$$

Using relations (6) and (5) and taking into consideration that the statical fields is constant ($\delta M_x = 0$, $\delta r = 0$), after some algebra one can obtain the desired first order variations

$$P_a\, \delta v_o + M_a\, \delta\, \psi_o = \int_o^L (M_{x,\bar{s}}\, v_a'' - r_{,\bar{s}}\, v_a)\delta\bar{s}\, dz \qquad (7)$$

If $P_a = 0$, $M_a = 1$ is assumed, then the first variation $\delta v_o'$ is determined, and if $P_a = 1$, $M_a = 0$, then the first variation δv_o is established.
In the case of the linear beam resting on the linear Winkler–type foundation, we arrive at

$$P_a\, \delta v_o + M_a\, \delta\psi_o = -\int_o^L [(EI)_{,\bar{s}}\, v'' v_a'' + k_{,\bar{s}}\, v v_a]\delta\bar{s}\, dz \qquad (8)$$

The derivation of the first variation of the internal forces proceeds in a similar manner.
The distortions

$$\Delta\psi_{oa} = \psi_{oa}\,(z = z_o^+) - \psi_{oa}\,(z = z_o^-) \text{ and}$$
$$\Delta v_{oa} = v_{oa}\,(z = z_o^+) - v_{oa}\,(z = z_o^-) \qquad (9)$$

are imposed on the same adjoint beam. Consider some kinematically admissible variations of static field of the primary beam.
According to the virtual work theorem one can write

$$\delta M_o\, \Delta\psi_{oa} + \delta T_o\, \Delta v_{oa} = -\int_o^L (v_a''\, \delta M_x - v_a\, \delta r) dz \qquad (10)$$

Inserting relation (6) in (10) and taking into account that there are no external loads acting at the adjoint beam, one can express the first variation of the internal forces in terms of the design variable variations

$$\delta M_o\, \Delta\psi_{oa} + \delta T_o\, \Delta v_{oa} = -\int_o^L (M_{x,\bar{s}}\, v_a'' - r_{,\bar{s}}\, v_a)\, \delta\bar{s}\, dz \qquad (11)$$

Obviously in the case of a linear structure one can obtain

$$\delta M_o\, \Delta\psi_{oa} + \delta T_o\, \Delta v_{oa} = \int_o^L [(EI)_{,\bar{s}}\, v'' v_a'' + k_{,\bar{s}}\, v v_a]\, \delta\bar{s}\, dz \qquad (12)$$

4. NUMERICAL EXAMPLES

Consider for example a pile made of a linear material with Young's modulus E resting on a linear, uniform Winkler–type foundation and subjected to a unit load P = 1 and a unit moment M = 1 at the pile top, as shown in Fig. 1. Utilizing relation (7), the first variations of the horizontal displacement δv_o and the angle of cross–section rotation $\delta\psi_o$ at the pile top can be expressed in terms of the flexural stiffness variation δ(EI) and the foundation stiffness variation δk

$$\delta v_0 = \int_0^L (F_v^P + F_v^M)\delta k \, dz + \int_0^L (K_v^P + K_v^M)\delta(EI)dz$$

$$\delta\psi_0 = \int_0^L (F_\psi^P + F_\psi^M)\, \delta k \, dz + \int_0^L (K_\psi^P + K_\psi^M)\delta(EI)dz \tag{13}$$

The reciprocity theorem (Washizu, 1974) allows us to show that $F_v^M = F_\psi^P$ and $K_v^M = K_\psi^P$. The static analysis of the beam resting on the Winkler-type elastic foundation necessary to find the distribution of the underintegral coefficients is carried out by the finite element method. It is worthwhile noticing that the pile is considered to be a long one (1) because of the ratio of the length L to the characteristic length of the soil-pile system T(L/T = 5). The distributions calculated are shown in Fig. 2 ÷ 3.

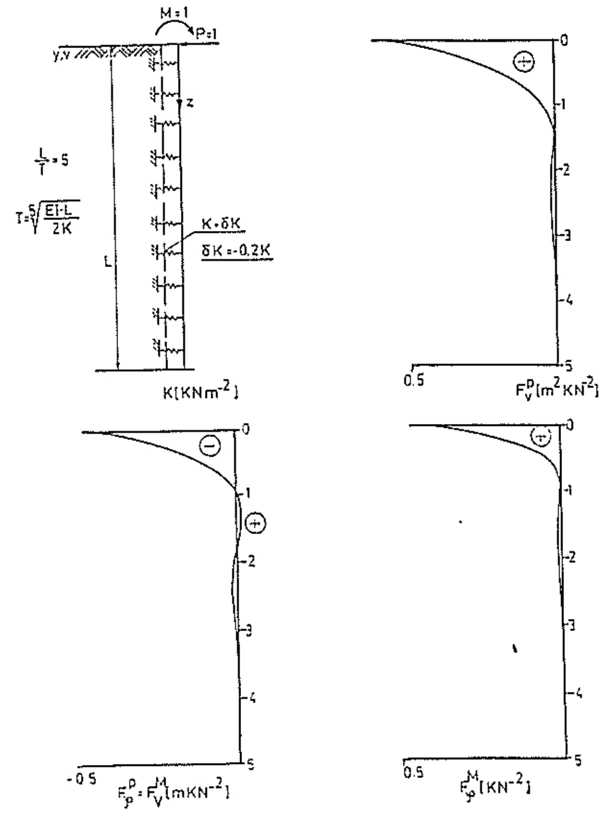

Figure 2. Distributions of underintegral coefficients of foundation stiffness variation δk for uniform foundation stiffness

To estimate the approximation of the change of the displacements due to 20% reduction of the design variable variations by virtue of their first variations, the horizontal displacement and the angle of cross-section rotation at the pile top are determined for the pile with the changed design variation and compare to their approximated values. The results obtained and the errors of approximations are presented in Table I.

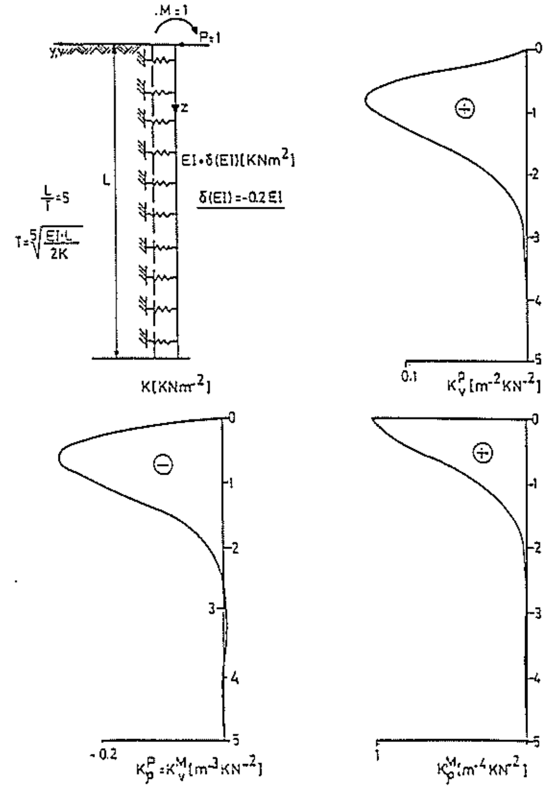

Figure 3. Distributions of underintegral coefficients of flexural stiffness variation of the pile for uniform foundation stiffness

5. CONCLUSIONS

In the paper the first order variations of arbitrary displacements and internal forces of laterally loaded pile due to the design variable increments are derived. The sensitivity analysis discussed is valid for both linear and nonlinear behaviour of the pile material and the soil. The first variations derived enable us to calculate the change of the displacements and the internal forces due to the increments of the pile cross-section dimensions, the pile material constants and the soil constants without full reanalysis of the pile. The latter may be especially interesting for the structures situated in cold regions, where the soil constants may vary due to freezing and thawing effects.

The results presented allow us to draw conclusion that approximation of the changes of the displacements of the pile by means of their first variations is good even for 20 per cent change of the design variable.

6. ACKNOWLEDGEMENT

The financial support of the Natural Science and Engineering Research Council of Canada under grant number OGP 110262 is gratefully acknowledged.

TABLE I

COMPARISON OF EXACT AND APPROXIMATE VALUES OF THE LATERAL DISPLACEMENT AND THE ANGLE OF CROSS-SECTION ROTATION AT THE PILE TOP FOR 20% REDUCTION OF THE DESIGN VARIABLES

	DESIGN VARIABLE (S)			
	FLEXURAL STIFFNESS		FOUNDATION STIFFNESS	
	LOADINGS			
	P = 1	M = 1	P = 1	M = 1
$v_0(s)$	0.71177	–0.63310	0.71177	–0.63310
$v_0(0.8\ s)$	0.84153	–0.70807	0.75249	–0.70759
approx. v_0 (0.8 s)	0.81861	–0.69664	0.74732	–0.69628
Error [%]	2.72	1.61	0.69	1.60
ψ_0 (s)	–0.63310	1.12573	–0.63310	1.12573
ψ_0 (0.8 s)	–0.70807	1.19100	–0.70757	1.33024
approx. ψ_0 (0.8 s)	–0.69664	1.18260	–0.69628	1.29682
Error [%]	1.61	0.71	1.60	2.51

7. REFERENCES

Das, B.M. (1990). Principles of Foundation Engineering. PWS–Kent Publ. Comp., Boston.

Desai, C.S. and Kuppusamy, T. (1980). Application of a Numerical Procedure for Laterally Loaded Substructures. In Numerical Methods of Offshore Piling, Proceedings of a Conference, Institution of Civil Engineers, London, pp. 93–100.

Mroz, Z., Kamat, M.R. and Plant, R.H. (1985). Sensitivity Analysis and Optimal Design of Nonlinear Beams and Plates. J. Struct. Mech., Vol. 13, pp. 245–266.

Szefer, G., Mroz, Z. and Demkowicz, L. (1987). Variational Approach to Sensitivity Analysis in Nonlinear Elasticity. Arch. Mech. Vol. 39, pp. 247–259.

Washizu, K. (1974). Variational Methods in Plasticity and Elasticity. Pergamon Press, Oxford.