A Point Estimate Method for Calculating the Reliability Index of Slopes

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SUMMARY: A point estimate method is developed for calculating the reliability index of slopes. The new point estimate method has the same order of accuracy as Rosenblueth's method which is commonly used for geotechnical reliability analysis, but the proposed method is much more efficient.

1. INTRODUCTION

In current probabilistic approaches to slope analysis, the safety of a slope is usually measured by the reliability index β which is defined as

$$\beta=\frac{\mu_F}{\sigma_F} \tag{1}$$

where μ_F and σ_F are respectively the mean value and standard deviation of the factor of safety F. The usual approach for calculating μ_F and σ_F is based on the Taylor's series expansion about the mean values of the random variables (1). This method requires the calculation of derivatives of F. Very often, difficulties arise in obtaining these derivatives because the factor of safety function $F(\underline{X})$, where X is the collection of random input parameters, is either too complex or it cannot be expressed in an explicit mathematical form (e.g. $F(X)$ may represent a program which computes the factor of safety of a slope for a given set of input parameters X). In this case, it would be convenient to use a point estimate method (PEM) which does not require the calculation of derivatives.

This paper reviews the existing point estimate methods and proposes a new PEM which is much more efficient than the PEM developed by Rosenblueth (25),(26) commonly used for probabilistic analysis of slopes.

2. PROBABILISTIC MODELLING OF SOIL PROPERTIES

Before reviewing the existing PEMs, it is necessary to discuss the existing approaches to probabilistic modelling of soil properties because the misuse of PEM in some of the existing publications on probabilistic analysis of slopes is related to incorrect modelling of soil properties. In probabilistic analysis of slopes, an approach exists in which the random variation of a soil property within the soil profile is represented by one single random variable (19),(20),(21),(22),(24),(29),(30). This approach is referred to as the single-random-variable (SRV) approach in the present paper. As illustrated in Fig.1(a), the SRV approach implies that all realisations of the soil property are the same at all location, although the exact magnitude of the realisation remains random based on this model. As pointed by the author in many occasions (6),(10),(12),(13), this approach is erroneous because if the model is correct, the actual magnitude of the soil property at all locations within the soil profile can be determined with absolute certainty by measuring the value of the soil property using one sample taken anywhere within the profile. Unfortunately, the SRV approach is still being used for probabilistic analysis of slopes.

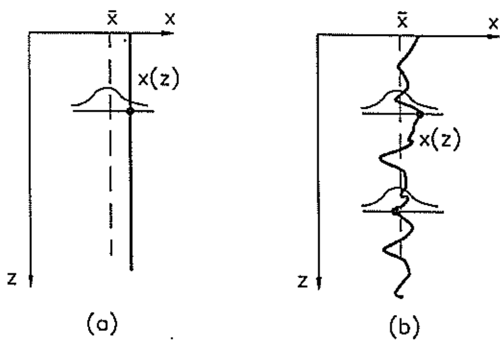

Fig.1 Probabilistic modelling of soil profile

A more realistic approach is to model a soil profile as a random field as shown in Fig.1(b). In this model, the values of the soil property at different locations are treated as separate random variables. Although the concept of random field has been used for a long time in mining engineering (18), it has only been introduced to soil engineering in the early 1970s, notably by Cornell (3), Lumb (16),(17) and by Vanmarcke (28).

The use of the SRV approach will result in a gross overestimation of the failure probability of slopes (8),(12),(13),(15) because it ignores the variance reduction due to spatial averaging. This can be discerned by a simple example of a homogeneous cohesive slope with a planar slip surface. The shear resistance R along the slip surface of length L can be expressed as $R=L{\cdot}c_L$ where c_L is the spatial average of the undrained shear strength c_u along the slip surface. It is assumed that c_u has a constant mean trend versus depth as shown in Fig.1.

If the SRV approach is used, c_L is equal to c_u because the undrained shear strength has the same magnitude everywhere within the soil profile. Therefore $\mathrm{var}\{c_L\}=\sigma^2$ where σ^2 is the variance of c_u. In reality, the variation of c_u is similar to that shown in Fig.1(b). A low value of c_u at one location is compensated by a high value at another location along the slip surface. As a result, the variability of the spatial average is less than that of the point property due to the compensating effect. The variance of c_L can be expressed as $\sigma^2 \cdot \Gamma^2(L)$ where $\Gamma^2(L)$ is the variance reduction factor which can be approximated by δ/L (28). δ is the scale of fluctuation. It is a measure of the spatial extent within which a soil property exhibits a strong autocorrelation.

In the SRV approach, soil properties are assumed to be perfectly correlated because the values of a soil property are the same at all locations within the soil profile. This corresponds to an infinitely large scale of fluctuation. In practice, the scale of fluctuation is typically of the order of a few metres (14). Therefore, the variance reduction due to spatial averaging is very significant. For instance, if δ=2m, the variance reduction factor for a 10m long slip surface is about 0.2. The variance of the spatial average is therefore only one-fifth of the value based on the SRV approach.

The SRV approach grossly overestimates the variability of spatial averages of soil properties, giving an unrealistically high failure probability. For instance, Matsuo & Kuroda (19) reported failure probabilities of slopes of the order of 20 to 40% for the typical range of factor of safety of 1.2 to 1.4.

The random field model which gives a more realistic prediction of failure probability of a slope is now well accepted by many researchers as a useful model for probabilistic modelling of soil profiles (1),(8),(12),(15),(31). Unfortunately, the SRV approach is still being used by soil engineers who are unaware of the random field model or by some researchers who ignore the development of the more realistic albeit slightly more difficult random field model. The continued use of the SRV approach gives practitioners, such as Peck (23), the feeling that the discipline of geotechnical reliability has not reached maturity because the high failure probability predicted by this approach is obviously incompatible with the observed frequency of failures.

2. POINT ESTIMATE METHODS

Point estimate methods refer to those methods which enable the statistical moments of a random function $f(\mathbf{X})$ to be calculated using the values of $f(\mathbf{X})$ at a specific set of values of $\mathbf{X}$. An accurate and efficient PEM has been developed by Evans (4),(5), and also discussed in (11) and (16), for functions with independent random variables. Although Evans' method is accurate, it is of limited use in geotechnical reliability analysis because soil properties are spatially autocorrelated and therefore not statistically independent.

Rosenblueth (25),(26) developed a 2-point PEM for correlated random variables in which the joint probability density of $\mathbf{X}$ is assumed to be concentrated at points in the 2^n hyperquadrants of the space defined by $\mathbf{X}$. The expected value of $f(\mathbf{X})$ is obtained by summing the product of $f(\mathbf{X})$ and the probability content of $\mathbf{X}$ for all $\mathbf{X}$ in the 2^n hyperquadrants. Rosenblueth's method involves 2^n evaluations of $f(\mathbf{X})$. As the number of evaluations increases exponentially with n, the method becomes impracticable when n is large.

Rosenblueth's method is commonly used for probabilistic analysis of slopes by researchers who adopt the SRV approach (20),(21),(22). If the SRV approach is used, the number of random variables will be small and so will be the number of evaluations of F required for the implementation of Rosenblueth's method. However, if one accepts the fact that soil profiles should be modelled as random fields, the number of random variables involved in slope stability analysis will be very large. Consider a slope which is analysed by the method of slices. If the soil properties are modelled as random fields, the spatial averages of strength parameters c', $\tan\phi'$ and the soil density γ for each slice should be modelled as separate random variables, giving a least $3n$ random variables for n slices. If there are 10 slices, the use of Rosenblueth's 2-point method will then require at least 2^{30} or 1704 million evaluations of F. This is hardly feasible in practice.

3. A NEW POINT ESTIMATE METHOD

A new PEM has recently been developed by Li (7). The new PEM has the same order of accuracy as Rosenblueth's 2-point for multivariate functions, but it is more efficient. It enables the reliability index of a slope to be calculated using a much smaller number of evaluations of F. The fomulation of the new PEM presented herein is different to that described in Li (7), but the final result is the same.

Given the first- and second-order statistical moments of the random variables, an exact expected value can only be obtained if $f(\mathbf{X})$ is of at most a quadratic polynomial which can be expressed in the following general form.

$$f(\mathbf{X})=f(\mu)+\sum_i a_i(x_i-\mu_i)+\sum_i b_i\cdot(x_i-\mu_i)^2+\sum_{ij} c_{ij}(x_i-\mu_i)\cdot(x_j-\mu_j) \tag{2}$$

where μ_i is the mean value of x_i and $\mu=(\mu_1,\mu_2,..,\mu_n)$ for a function of n variables. Taking expectation on both sides of (2), we obtain

$$\mathrm{E}\{f(\mathbf{X})\}=f(\mu)+\sum_i b_i\cdot\sigma_i^2+\sum_{ij} c_{ij}\cdot\sigma_i\cdot\sigma_j\cdot\rho_{ij} \tag{3}$$

where σ_i is standard deviation of x_i and ρ_{ij} is the correlation coefficient between x_i and x_j.

Define $f(..,\pm1,..,\pm1,..)=f(..,\mu_i\pm\sigma_i\ ,..,\mu_j\pm\sigma_j\ ,..)$, it can be proved that

$$f(\mu)+\sum_i b_i\cdot\sigma_i^2=\frac{1}{2^n}\sum_{p=0}^{1}..\sum_{q=0}^{1}....\sum_{s=0}^{1}f((-1)^p,..,(-1)^q,...,(-1)^s) \tag{4}$$

$$c_{ij}\cdot\sigma_i\cdot\sigma_j=\frac{1}{2^n}\sum_{p=0}^{1}\cdot\underbrace{\sum_{q=0}^{1}}_{i\text{th term}}..\underbrace{\sum_{r=0}^{1}}_{j\text{th term}}\cdot\sum_{s=0}^{1}\eta\cdot f((-1)^p,..,(-1)^q,..,(-1)^r.,(-1)^s) \tag{5}$$

where η=1 if $q+r$ is even and η=−1 if $q+r$ is odd. Substitution of (4) and (5) into (3) yields the 2-point estimates developed by Rosenblueth (25) for multivariate functions. The procedure described in (4) and (5) is not an efficient way for obtaining the different terms in (3). It requires a total of 2^n evaluations of $f(\mathbf{X})$. A more efficient PEM can be obtained using the following procedure.

Define $f_i(x_i)=f(\mu_1,..,\mu_{i-1},x_i,\mu_{i+1},..,\mu_n)$. According to (2), $f_i(x_i)$ can be expressed as:

$$f_i(x_i)=f(\mu)+a_i\cdot(x_i-\mu_i)+b_i\cdot(x_i-\mu_i)^2 \tag{6}$$

Since $f_i(x_i)$ is a quadratic function in x_i, application of the 2-point estimate developed by Rosenblueth (25) for a univariate function to (6) will yield the exact expectation of $f_i(x_i)$. Therefore,

$$\mathrm{E}\{f_i(x_i)\}=\frac{1}{2}\{f_i(x_i^+)+f_i(x_i^-)\}=f(\mu)+b_i\cdot\sigma_i^2 \tag{7}$$

where $x_i^+=\mu_i+\sigma_i$, $x_i^-=\mu_i-\sigma_i$. Summing (7) for all i, we obtain

$$f(\mu)+\sum_i b_i\cdot\sigma_i^2=(1-n)\cdot f(\mu)+\frac{1}{2}\sum_i [f_i(x_i^+)+f_i(x_i^-)] \tag{8}$$

Eq.(8) gives the first two terms on the right hand side of (5). To obtain the last term of (3), it is necessary to calculate the coefficient c_{ij}. It can be easily proven that $c_{ij}=\partial^2 f(\mathbf{X})/\partial x_i \partial x_j$ for a quadratic function. This derivative can be obtained by the following finite difference formula.

$$c_{ij}=\frac{f_{ij}(x_i^+,x_j^+)-f_i(x_i^+)-f_j(x_j^+)+f(\mu)}{\sigma_i\cdot\sigma_j} \tag{9}$$

where $f_{ij}(x_i^+,x_j^+)=f(\mu_1,\ldots,\mu_{i-1},x_i^+,\ldots,\mu_{j-1},x_j^+,\ldots,\mu_n)$. By substituting the results of (8) and (9) into (3), the following formula is obtained.

$$\begin{aligned}E\{f(X)\}=&(1-\frac{3n}{2}+\frac{\rho}{2})\cdot f(\mu)+\frac{1}{2}\sum_i [(3-2\cdot\rho_i)\cdot f_i(x_i^+)+f_i(x_i^-)]\\&+\sum_{i<j} f_{ij}(x_i^+,x_j^+)\cdot\rho_{ij}\end{aligned} \tag{10}$$

where $\rho_i=\Sigma_j\ \rho_{ij}$, $\rho=\Sigma_i\ \rho_i$ and $\rho_{ii}=1$ by definition. Eq.(10) only requires $(n^2+3n+2)/2$ evaluations of $f(\mathbf{X})$ as compared to 2^n evaluations for Rosenblueth's 2-point PEM. The proposed PEM is more efficient than Rosenblueth's method for $n>3$ and is much more so when n is large. If the random variables are independent, Eq.(10) can be simplified to the following PEM which only requires $2n+1$ evaluations of $f(\mathbf{X})$.

$$E\{f(\mathbf{X})\}=(1-n)\cdot f(\mu)+\frac{1}{2}\sum_i [f_i(x_i^+)+f_i(x_i^-)] \tag{11}$$

If the calculation of $f(\mathbf{X})$ is time consuming, it is preferable to transform the random variables $\mathbf{X}$ into a new set of uncorrelated random variables, say $\mathbf{Z}$, by means of an orthogonal transformation. In this case, the expected value of $f(\mathbf{X})$ can be obtained by applying (11) to the function in the transformed parameter space $\mathbf{Z}$ (8).

5. ILLUSTRATIVE EXAMPLE

The example used in this study is shown in Fig.2. Both the SRV approach and the random field model are used. The mean value of F can be calculated using the proposed PEM by taking $f(\mathbf{X})=F$. Similarly, the variance and hence the standard deviation of F can be obtained by taking $f(\mathbf{X})=(F-\mu_F)^2$. The reliability indices obtained from these two approaches are compared.

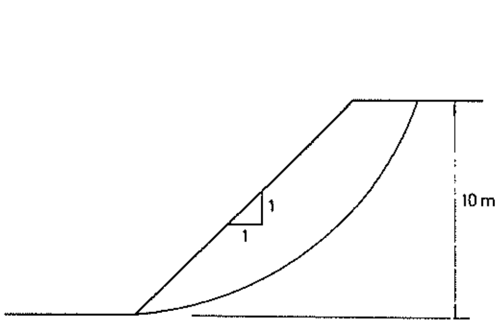

Fig.2 Details of illustrative example

The input parameters are shown in the following table. For simplicity the cross-correlation of soil properties is neglected, i.e. cov{c',tanϕ'}=0 and so on.

Soil property	mean value	coeff. of variation
c'	8 kPa	20%
tanϕ'	tan30°	10%
γ	18 kN/m^3	5%

When using the random field model, the following autocorrelation function is assumed for all the soil properties.

$$\rho(\tau_x,\tau_y)=e^{-0.4\cdot(\tau_x+\tau_y)}$$

where τ_x and τ_y are respectively the lag distances in the horizontal and vertical directions in metres. An approximate Morgenstern and Price's method based on the condition of overall moment equilibrium is used, assuming a constant interslice function and a value of $\lambda=0.6$. The moment is taken about point A in Fig.2. The method of analysis is described in detail elsewhere (12),(13),(15).

The reliability indices based on the SRV approach and the random field model are 1.882 and 3.144 respectively. The SRV approach gives a much lower reliability index of the slope. This shows clearly the importance of using the correct approach for modelling the spatial variability of soil properties. The reliability index based on the SRV approach can also be obtained using Rosenblueth's method due to the small number of variables involved. The result is 1.880 which is very close to the value obtained from the proposed PEM. This is to be expected because the PEM based on (10) has the same order of accuracy as Rosenblueth's method.

6. CONCLUSIONS

The limitations of the existing point estimate methods are discussed. It is observed that the commonly used 2-point PEM developed by Rosenblueth is too efficient for use in geotechnical reliability analysis. A more efficient point estimate method is proposed for calculating the reliability of slopes.

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