Some Applications of Fuzzy Mathematics to Rock Engineering and Slope Stability

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SUMMARY In this paper, the basic concepts of fuzzy mathematics are described and applied to several important problems in rock engineering. A method is presented for determining the fuzzy values of typical rock mass parameters. The limit equilibrium method is used as the basis for calculation of the stability factor for a rock slope and fuzzy arithmetical operations are required to evaluate this factor because the parameters used in the calculation are considered to be indistinct. The techniques presented in the paper are illustrated by typical example calculations.

1. INTRODUCTION

Many methods of analysis have been formulated to allow quantification of the degree of stability of structural systems. In rock slope engineering the predominant technique is the limit equilibrium method. However, most of these analyses are strictly deterministic in context, i.e. they assume that the structure and its environment are deterministic quantities. In most of these analyses, a criterion for structural stability is generally established. Suppose a state parameter K is defined as a function of uncorrelated random variables $x_1, x_2, \ldots, x_n$, all of which have an influence on structural stability. The equation

$$K(x_1, x_2, \ldots, x_n) = 0 \quad (1)$$

is called the limit state condition. For example, if we consider the simple problem of a sliding block and assume that K is a function of a resistance force R and a sliding force S, both acting in the direction of sliding, then the limit state equation can be written as

$$K(R,S) = R - S = 0 \quad (2)$$

Thus the structure will be stable when $K \geq 0$ and, conversely, failure will occur when $K < 0$.

Although this is essentially a deterministic representation of stability, some parameters involved in the calculation may be "inexact" or "indistinct", in that their precise values may be difficult to determine. In other words, uncertainties exist in most aspects of structural stability, particularly when such problems occur in nature. Loads, environmental factors, material properties, and structural dimensions are usually difficult to predict due to lack of sufficient data. Furthermore, knowledge of the full complexity of the problem is often imperfect, i.e. some of the mechanisms that control stability may be poorly understood, which gives rise to additional uncertainty.

The ratio of the resisting force to the sliding force along the failure plane is called the stability factor. Thus this conventional stability factor, which is obtained from appropriate calculation, can be considered to be "fuzzy", e.g. although the calculated value of a stability factor may be greater than one, there is a finite probability that its true value may be less than one. It is because of this "fuzziness" that the stability number should not be called a safety factor.

It is worthwhile to pursue a method of analysis which can adequately describe the "fuzziness" for this and other problems in rock engineering. Fuzzy mathematics (Zadeh, 1965) has provided a potential tool for solving many real engineering problems and to date it has been used successfully on a number of important practical problems (e.g. Nguyen, 1985; Boissonnade, 1986; Xiao & Zhou, 1987; Kacewicz, 1987; Sakurai & Shimizu, 1987; Xiao & Yu, 1989; 1990).

In this paper the basic techniques of fuzzy mathematics are described. The method is used to evaluate the stability of rock slopes where potential failures are determined in each case by a single joint plane on which sliding of a rock block may occur. The use of the method to describe the fuzziness of one of the rock mass strength parameters is also illustrated.

2. BASIC CONCEPTS OF FUZZY MATHEMATICS

In this section some basic definitions and some of the important concepts of fuzzy mathematics are presented for completeness. More details are discussed by Zadeh (1965, 1975). Where appropriate, the important concepts are illustrated by examples.

2.1 Definition of a Fuzzy Set

A fuzzy set of objects x is defined as a set of ordered pairs, i.e.

$$I = \{ \mu(x) , x \} \quad (3)$$

where $\mu(x)$ is termed the "grade of membership of x in I". $\mu(x)$ may only take values in the closed interval [0,1]. For example, in structural problems, if stability is almost certain to occur, then $\mu(K)$ is equal to 1. This means that the membership function $\mu(K)$ can be regarded as a linguistic variable. In reality, there are many cases in which the transition from membership to non-membership of an object in a set is gradual rather than abrupt. For such cases the grade of membership of x in I is represented by values of $\mu_I(x)$ in the range from 0 to 1. If the fuzzy set I contains a finite number of members, then it can also be expressed as:

$$I = \mu_I(x_1)/x_1 \cup \mu_I(x_2)/x_2 \cup \ldots \ldots \cup \mu_I(x_n)/x_n = \int \mu_I(x)/x \tag{4}$$

The symbol $\bigcup$ is used to represent the union operation, and the symbol / denotes the correspondence between an object in the set and its membership function. $\int$ is used here to represent all relationships between elements of the fuzzy set I and their degrees of membership. $\int$ is **not** used here to denote integration.

2.2 Fuzzy Number

A fuzzy number is a quantity that is characterised by a distribution (either discrete or continuous) of possible values. For example, consider the case where the cohesion c of a rock mass can be regarded as a fuzzy number, which is expressed as

$$c = 0.8/95 \bigcup 1.0/100 \bigcup 0.7/105.$$

In loose terms, this is the same as stating that c is "approximately 100". More formally, the above expression means that the cohesion has several possible discrete values, viz. a value of 100 with a membership grade of 1.0, a value of 95 with a grade of 0.8, and value of 105 with a grade of 0.7. Thus the membership function expresses the likelihood that the parameter has the nominated value.

2.3 Extension Principle

One of the basic ideas of fuzzy set theory, which provides a general extension of nonfuzzy mathematical concepts to fuzzy environments, is the extension principle. Consider a function f that provides a mapping of the real number x onto y, i.e. $f: x \rightarrow y$. This mapping concept can also be applied to fuzzy numbers, i.e. if I is fuzzy number, then $f: I \rightarrow f(I)$. This is called an "extension" of the mapping f.

From the extension principle, two inferences may be obtained and these are discussed below.

2.4 Inference 1

If C is a constant, then

$$C*\int \mu_I(x)/x = \int \mu_I(x)/C*x \tag{5}$$

where * denotes one of the arithmetical operations of multiplication, addition, division or subtraction. In other words, the first inference of the extension principle is that any arithmetic operation on a fuzzy set implies that the same operation is carried out on all elements of the set, but the values of the membership function for the newly formed set are the same as those of the original set.

2.5 Inference 2

If f is a relationship or a monotonic function that provides a mapping from x to y and I is a fuzzy set expressed as

$$I = \int \mu_I(x)/x \tag{6}$$

then

$$f(\int \mu_I(x)/x) = \int \mu_I(x)/f(x) \tag{7}$$

This equation states that the image of I under the relationship f can be deduced from the knowledge of the images of x under f.

2.6 Theorem

If I, J and K are three fuzzy numbers and their membership functions are μ_I, μ_J and μ_K respectively, and if $K = I * J$ is the result of an arithmetic operation on the fuzzy numbers I and J, then the membership function of K is given by

$$\mu_K(z) = \vee (\mu_I(x) \wedge \mu_J(y)) \tag{8}$$

where $\vee$ is the symbol used to indicate that the maximum should be selected from all possible values of the membership function, and $\wedge$ is the symbol used to indicate that the minimum should be selected from the possible values. Zero must not be a possible value of the fuzzy number J whenever * represents the operation of division.

Example

An example is now presented to illustrate some of the definitions and concepts discussed above. Suppose fuzzy numbers I and J are given as follows

$$I = 0.7/3 \bigcup 1.0/2 \bigcup 0.8/1$$

$$J = 0.9/3 \bigcup 0.6/2$$

then the operation of addition of these two fuzzy numbers can be represented as

$$\begin{aligned} I + J &= (0.7 \wedge 0.9)/(3+3) \bigcup (1.0 \wedge 0.9)/(2+3) \bigcup \\ &\quad (0.8 \wedge 0.9)/(1+3) \bigcup (0.7 \wedge 0.6)/(2+3) \bigcup \\ &\quad (1.0 \wedge 0.6)/(2+2) \bigcup (0.8 \wedge 0.6)/(2+1) \\ &= 0.7/6 \bigcup (0.9 \vee 0.6)/5 \bigcup (0.8 \vee 0.6)/4 \bigcup 0.6/3 \\ &= 0.7/6 \bigcup 0.9/5 \bigcup 0.8/4 \bigcup 0.6/3 \end{aligned}$$

3. DETERMINATION OF FUZZY PARAMETERS

One of the most difficult problems in applying the above techniques in practice involves determining the membership function for the parameters that are considered to be fuzzy. This problem arises in most applications, including the stability analysis of rock slopes. In this case the parameters which are important include those defining the geometry of the rock slope, its geological features such as jointing and bedding planes, and the shear strength of the planar features upon which rock blocks may slide. Data defining these "properties" are usually determined from field observation and laboratory testing or are estimated on the basis of experience. Discrete measurements or estimates of the important parameters are usually provided and more than one measurement of each parameter may be available. Since the data which are obtained from testing or observation may take a wide range of values, it is convenient first to normalise the primary data before proceeding to the determination of the membership function. Furthermore, it may be possible to measure or determine values for the design parameters in a variety of different ways. It is therefore desirable to have a rational means of assigning weights to the parameter values, based upon their method of determination. Suitable processes for normalising the data and for rationally selecting the weighting factors are considered below.

3.1 Normalisation of Data

Let $X = \{x_1', x_2', \ldots, x_m'\}$, where X is a set of data to be normalised. Normalisation of the data is carried out according to the following formula

$$x_i'' = \frac{x_i' - \overline{x'}}{s} \tag{9}$$

where the mean of sample X containing m values is given by

$$\overline{x'} = \left(\frac{1}{m}\right) \sum_{i=1}^{m} x_i' \tag{10}$$

and its standard deviation is

$$s = \sqrt{\frac{1}{m}\sum_{i=1}^{m} (x_i' - \overline{x'})^2} \tag{11}$$

If normalised data is required in the range [0,1], the extreme value standardisation formula may be employed, i.e.

$$x_i = \frac{x_i'' - Min\{x_i''\}}{Max\{x_i''\} - Min\{x_i''\}} \tag{12}$$

where $Max\{x_i''\}$ and $Min\{x_i''\}$ denote the maximum and minimum values, respectively, of the set of values x_i''.

Example

Suppose there are five measurements of the one parameter, i.e. $x_1' = 12.2, x_2' = 7.5, x_3' = 7.0, x_4' = 6.7$, and $x_5' = 6.1$. The mean and standard deviation of this set of data can be computed from equations (10) and (11) as 7.9 and 2.2, respectively. The values of x_i'' are computed using equation (9), i.e. $x_1'' = 1.95$, $x_2'' = 0.18$, $x_3'' = -0.41$, $x_4'' = -0.55$ and $x_5'' = -0.82$. Hence, the sample data can be normalised onto the closed interval [0,1] using equation (12) to give: $x_1 = 1$, $x_2 = 0.23$, $x_3 = 0.15, x_4 = 0.1$ and $x_5 = 0$.

3.2 Weighting Factors

In many practical problems it may be possible to obtain estimates of any one design parameter by a variety of means. As mentioned above, it is desirable to have a rational means of determining weights to be assigned to each of the measurements, according to the "reliability" of each of the different methods of assessment. If one method is more important or known to be more reliable than others, then it should be assigned a large weight. Conversely, if sometimes a method is known to produce doubtful or even spurious results, it should be assigned a low weight. A method for determining the weighting coefficients is now described.

If there are n measurements of a given parameter, then the $n \times n$ matrix V of influence coefficients is first established, i.e.

$$V = \begin{bmatrix} V_{11} & V_{12} & \ldots & V_{1n} \\ V_{21} & V_{22} & \ldots & V_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ V_{n1} & V_{n2} & \ldots & V_{nn} \end{bmatrix} \tag{13}$$

where the relative importance of the measurement or estimate m_i compared to m_j is quantified by the influence coefficient V_{ij}. Suggested values for V_{ij} are shown in Table I.

TABLE I

INFLUENCE COEFFICIENTS

Description	Values of V_{ij}
m_i is as important as m_j	1
m_i is only slightly more important than m_j	3
m_i is obviously important than m_j	5
m_i is much more important than m_j	7
m_i is overwhelmingly important than m_j	9

Note: Intermediate values of the influence coefficient may be defined, i.e. 2, 4, 6 or 8, as appropriate. If m_i is less important than m_j, then the reciprocals of the values of V_{ij} shown in Table I are used in the matrix V, i.e. $V_{ji} = 1/V_{ij}$ for $i \neq j$ and $V_{ii} = 1$.

Finally, the weight assigned to the factor m_i is defined by a weighting coefficient w_i, i.e.

$$w_i = \frac{\overline{w_i}}{\sum_{i=1}^{n} \overline{w_i}} \tag{14}$$

where

$$\overline{w_i} = \left[\prod_{j=1}^{n} V_{ij}\right]^{1/n} \tag{15}$$

Values of w_i will always be between 0 and 1 and the sum of the weights w_i will always be 1.

3.3 Membership Function

There exist many functions that could be used to define the membership of a fuzzy number. In this paper, the form of the membership function for problems in rock mechanics is suggested by the normal distribution used in probability theory. An important difference here is that the membership function is "weighted" according to the method by which the fuzzy parameter was determined. Accordingly, the membership function of the fuzzy parameter x_i is suggested as follows

$$\mu(x_i) = e^{-\xi \frac{(\bar{x} - x_i)^2}{w_i}} \tag{16}$$

where the mean of the values is determined by

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{17}$$

TABLE II

ROCK MASS PARAMETERS AND CSIR ROCK MASS RATINGS FOR MARBLE
(After China Nonferrous Metal Company and Daye Nonferrous Metal Company)

Parameters	c_i MPa	σ_c MPa	RQD	i	Condition of Joints	Ground water	Joint orientation
Values	19	79.4	70%	7	-	-	-
CSIR ratings	-	7	13	10	20	7	-15

Note: c_i is the cohesive strength of the intact rock,
σ_c is the uniaxial compressive strength of specimen,
RQD is a measure of the drill core quality,
i is the intensity of jointing, measured as the number of joints per metre.

and ξ is called "resolution number" whose value determines the "width" or "scale" of the membership distribution.

In general, a design parameter P can be expressed as a fuzzy number, as follows

$$P = \mu(x_1)/x_1 \cup \mu(x_2)/x_2 \cup \ldots \ldots \cup \mu(x_n)/x_n \quad (18)$$

There is thus some ambiguity about the value of P, and $x_1, x_2, \ldots, x_n$ are all possible values.

4. ILLUSTRATIVE EXAMPLES

To illustrate the use of the techniques described above, two different types of example are considered. In the first, the parameters that are commonly used to characterise the mechanical behaviour of a rock mass are discussed. In particular, a method to determine fuzzy values of cohesive strength of a rock mass are presented. In the second example, the problem of rock slope stability is addressed.

4.1 Example 1

In any discussion of rock parameters, it is important to distinguish characteristics of a specimen of intact rock from the properties of the rock mass as a whole. It is well known that the behaviour of the rock mass depends on the rock substance, the discontinuities as well as the presence of water and the existing in situ stress regime. At present, there are a number of methods which are used for determining rock mass strength properties on the basis of the strength of the intact rock material measured in the laboratory. For a given rock mass, different methods will result in different estimates of strength properties of the mass. Hence, in accordance with the concepts presented above, these parameters can be considered to be "fuzzy". As an example, it is possible to represent the cohesion of a rock mass by a fuzzy number, and procedures for doing this are presented below.

Consider a marble rock mass for which both field and laboratory data are available (China Nonferrous Metal company et al, 1987), as shown in Table II.

Using these data, at least four methods can be employed to estimate the cohesion of the rock mass which is denoted by c_m. These methods are described below.

(a) The CSIR geomechanics classification scheme for jointed rock masses (Hoek & Brown, 1980) may be used to provide an estimate of the rock mass cohesion. The data in Table II indicate that the overall rock mass rating (RMR) is 42, and so the rock mass can be described as Class III. For this class of rock mass the CSIR scheme suggests that the cohesion is likely to be in the range 150 - 200 kPa. For the present purpose a value of c_{m1} = 180 kPa has been selected as representative.

(b) The empirical strength criterion suggested by Hoek and Brown (1980), may be used to describe the strength of the rock mass, i.e.

$$\sigma_1 = \sigma_3 + \sqrt{m\sigma_c\sigma_3 + s\sigma_c^2} \quad (19)$$

Where σ_1 is the major principal stress, σ_3 is the minor principal stress applied to the specimen and m and s are constants which depend upon the properties of the rock and upon the extent to which it has been fractured before being subjected to the stresses σ_1 and σ_3,

As mentioned above, the total rating of the rock mass is 42. Hoek and Brown (1980) have suggested an approximate relationship between rock mass quality and the empirical strength constants. Based on this relationship the constants m and s are estimated as m=0.5 and s=0.0001.

Using the data in Table II, and the values of m and s deduced above, Mohr circles of stress corresponding to failure may be drawn. Hence the cohesion of the rock mass, applicable over the normal stress range from 0 to 10 MPa, can be determined graphically by fitting a straight line envelope to the failure circles. Following this procedure a value of c_{m2} = 220 kPa can be obtained.

(c) Hubbert et al (1960) have suggested a correlation between the intensity of rock jointing and the cohesion of the rock mass. This correlation is presented in Figure 1, where the ratio of rock mass cohesion, c_m, to the cohesion of the intact rock, c_i, is plotted against i, the "intensity" of the jointing. Based on this relationship and using the values of c_i and i presented above, a value for the rock mass cohesion is estimated as c_{m3} = 608 kPa.

(d) China Nonferrous Metal Company and the Daye Nonferrous Metal Company (1987) have deduced from

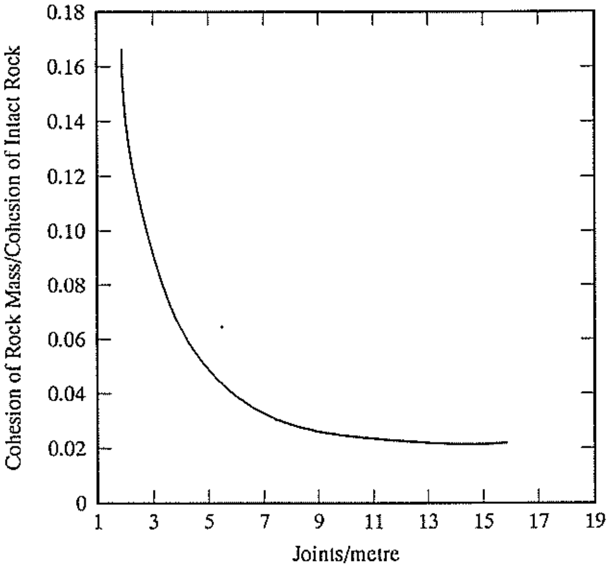

Figure 1 Relationship between joint spacing and cohesion of rock mass (after Hubbert et al)

their experience that a value of c_m may also be estimated from the formula

$$c_m = \frac{c_i}{(i+25)} \tag{20}$$

For the rock mass in question, equation (20) provides an estimate of c_{m4} = 594 kPa.

Each of these four estimates of the rock mass cohesion can be normalised onto the interval [0,1], using equations (10)-(13), to give: $c_{m1} = 0$, $c_{m2} = 0.094$, $c_{m3} = 1.0$ and $c_{m4} = 0.967$.

In this example, the first method for determining c_m is considered only slightly more important than the second, and much more important than the third, and obviously more important than the fourth. The matrix V, ranking the relative influence of each method, is established on the basis of Table I as follows:

$$V = \begin{bmatrix} 1 & 3 & 7 & 5 \\ 1/3 & 1 & 4 & 3 \\ 1/7 & 1/4 & 1 & 2 \\ 1/5 & 1/3 & 1/2 & 1 \end{bmatrix} \tag{21}$$

The weighting coefficients for each method can be obtained from equation (15), as $w_1 = 0.576, w_2 = 0.254$, $w_3 = 0.093$ and $w_4 = 0.077$.

Based on experience, it is suggested that a reasonable value of the resolution number ξ is 0.2 for this case. It should be noted, however, that a rational means for determining appropriate values of ξ for other types of problem have not been developed. This matter requires further research.

Substituting the normalised data, the weighting coefficients and the resolution number into equation (17), the membership grades of the fuzzy number c_m are obtained as $\mu(c_{m1}) = 0.94$, $\mu(c_{m2}) = 0.81$, $\mu(c_{m3}) = 0.60$ and $\mu(c_{m4}) = 0.59$. Therefore, the fuzzy number c_m can be expressed as follows:

$$c_m = 0.81/180 \cup 0.94/220 \cup 0.6/608 \cup 0.59/594 \text{ kPa}$$

4.2 Example 2

Consider now the problem of the sliding of a rock block on a planar sliding surface, as shown in Figure 2. H is the height of the slope, γ is the unit weight of rock mass, θ is the slope angle and β is the inclination of the potential failure plane. Gravity acts in the vertical direction. The shear strength of the planar interface is described by the Mohr-Coulomb criterion with a cohesive component of the shear strength c and friction angle φ.

Under the action of gravity, the sliding force per unit width acting along the failure plane is obtained from simple statics as:

$$F_s = \frac{LH\gamma}{2}(\cos\beta - \frac{\sin\beta}{\tan\theta})\sin\beta \tag{22}$$

where L is the length of failure plane.

The force resisting this sliding can be shown to be:

$$F_r = \frac{LH\gamma}{2}(\cos\beta - \frac{\sin\beta}{\tan\theta})\cos\beta\tan\phi + Lc \tag{23}$$

The slope stability factor is conventionally expressed as:

$$K = \frac{F_r}{F_s} = \frac{2c}{\gamma H(\cos\beta - \frac{\sin\beta}{\tan\theta})\sin\beta} + \frac{\tan\phi}{\tan\beta} \tag{24}$$

In this example, the parameters c, ϕ and β are all regarded as fuzzy numbers. They are defined as follows:

$c = 1/100 \cup 0.7/105$ kPa
$\phi = 1/35° \cup 0.8/37°$
$\beta = 1/45° \cup 0.5/43°$

The remaining parameters defining this problem are assumed to have "crisp" (non-fuzzy) values, i.e.

$$\gamma = 20 \text{ kN/m}^3, \quad H = 40 \text{ m}, \quad \theta = 60°$$

The stability factor may be calculated from equation (24), using Inferences 1, 2 and the theorem presented previously to perform the appropriate arithmetic operations on the fuzzy numbers. These operations give the fuzzy stability factor as:

$$K = 0.5/1.7 \cup 0.5/1.8 \cup 1.0/1.9 \cup 0.8/2.0 \cup 0.5/2.1 \cup 0.5/2.2$$

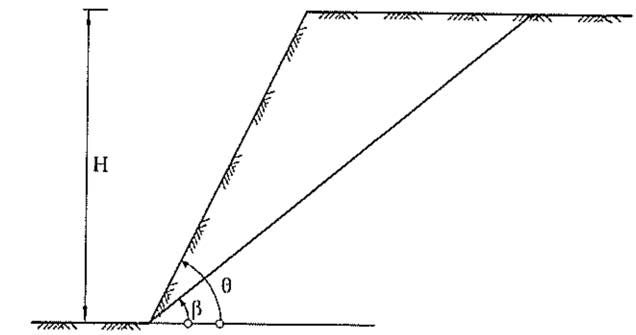

Figure 2 Geometry of the slope

TABLE III

QUALITATIVE DESCRIPTION OF THE MEMBERSHIP FUNCTION

Class	Value of Membership Function	Meaning
1	$\mu(K) = 0$	Certainly will not occur
2	$0 < \mu(K) \leq 0.1$	Very unlikely to occur
3	$0.1 < \mu(K) \leq 0.3$	Seldom occurs
4	$0.3 < \mu(K) \leq 0.5$	Could occur
5	$0.5 < \mu(K) \leq 0.7$	Likely to occur
6	$0.7 < \mu(K) \leq 0.9$	Very likely to occur
7	$0.9 < \mu(K) \leq 1.0$	Will almost certainly occur

In order to describe the possible distribution of the stability factor *K*, Table III provides suggested meanings that are to be associated with selected values of the membership function for *K*.

The relationship between *K* and its membership function is shown in Figure 3. This figure indicates that a stability factor of 1.9 has a value of the membership function equal to 1. According to Table III, this outcome can be described qualitatively as "will almost certainly occur". A value of the factor equal to 2.0 is "very likely to occur" and other values of the stability factor, viz. 1.7 to 1.8 and 2.1 to 2.2 "could occur".

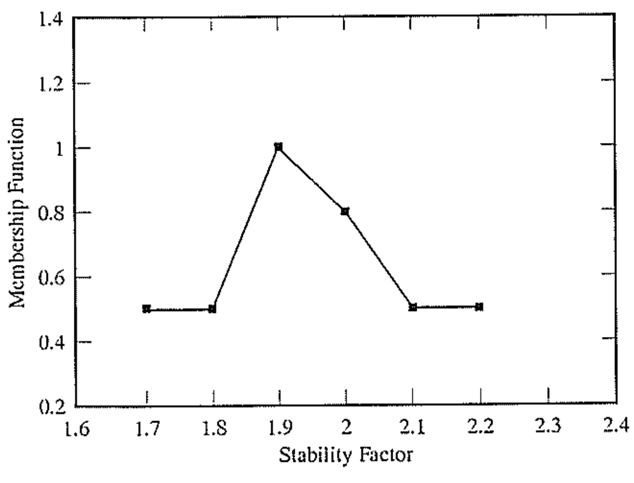

Figure 3 Fuzzy stability factor

5. CONCLUSION

The methods presented in this paper can be used not only to establish a model for describing complex or imperfect systems, but can also make full use of human ability to handle imprecise and vague information.

The procedures required to determine the fuzzy number in rock engineering and the fuzzy stability factor for a rock slope have been presented, together with the essential background theory of fuzzy mathematics.

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