Full Probabilistic Design Method of Slopes Considering Stratigraphic Uncertainty and Spatial Variability of Soil Parameters

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Abstract: Stratigraphic uncertainty is often ignored in slope reliability-based design, even though the spatial variability of soil parameters is considered. This paper proposes a full probabilistic design method for the slope considering these two types of variability. In the full probabilistic design framework, a generalized coupled Markov chain model is combined with a random field model to simultaneously characterize these two types of variability. The procedure for this method is presented. A slope is taken as an example for reliability-based design using the borehole data in Perth, Australia. To illustrate the importance of considering stratigraphic uncertainty and spatial variability of soil parameters in slope reliability-based design, the design results associated with two cases (i.e. only considering spatial variability of soil parameters and considering both types of uncertainty) are compared. The results indicate that the proposed method can effectively conduct slope reliability-based design considering these two types of variability. If only the spatial variability of soil parameters is considered, the design results mainly depends on the used stratigraphic distribution. If the geotechnical practitioners infer a stratigraphic distribution with a higher proportion of strong soil materials than the reality, the resulting optimal design scheme will lead to dangerous slope. In the opposite case, the resulting optimal design scheme will be conservative. To obtain the optimal design scheme accurately, the influences of these two types of uncertainty should be taken into account in slope reliability-based design.

Keywords: Slope; reliability-based design; stratigraphic uncertainty; spatial variability of soil parameters.

1 Introduction

There are many uncertainties involved in slope engineering, and these uncertainties have a significant impact on the deformation and stability of slope (Li et al. 2016; Deng et al. 2017; Qi and Liu 2019). Soil heterogeneity is one of the major sources of these uncertainties, which is roughly manifested in twofold: stratigraphic uncertainty and spatial variability of soil parameters (Elkateb et al. 2003). Traditionally, working stress design is often used in slope stability design. Although the concept of this method is simple and easy to understand by engineers, but it cannot effectively consider various uncertainties (Loehr et al. 2016). In recent years, Load and Resistance Factor Design (LRFD) has been paid more and more attention in geotechnical engineering field. This design method considers the uncertainty of load and resistance separately from the point of view of probability. It provides a more reasonable and economical design method for geotechnical structures (Paikowsky 2016; Rodrigo and Kim 2013; Pantelidis and Griffiths 2014). This design method has been adopted in the national building code of Canada (Becker 1997), European Code 7 (Orr 2012), and Japanese Geo-code 21 (Honjo and Kusakabe 2002). However, there are still three challenges in effectively applying the LRFD method. The first one is that the assumptions and/or simplifications adopted in the calibration processes are unknown to geotechnical practitioners. The second one is that, for retaining structures or slopes, the load and resistance are usually originated from the same sources (e.g. effective stress of soil) and correlated with each other. The last one is that it is difficult to consider the soil heterogeneity in the calibration process. How to solve these challenges in LRFD is also the current trend of research, which deserves a further study. However, these challenges can be easily addressed by a full probabilistic design. In the procedure of this design method, the reliabilities of possible designs can be evaluated using reliability analysis methods, and then the optimal design can be selected from it. The currently advanced reliability analysis methods can be used in this design procedure (Low and Phoon 2015). In recent years, this design method has been well applied in some complicated geotechnical problems. For example, Wu et al. (2012) adopts full probabilistic design method to the design of foundation pit engineering considering the spatial variability of undrained shear strength of soil; Wang and Cao (2013) develops an expanded reliability-based design (RBD) approach and applied it to the drilled shaft design; Li et al. (2016) further developed this design approach for efficient geotechnical RBD. However, the previous work have barely considered both stratigraphic uncertainty and spatial variability of soil parameters.

This paper aims to propose a full probability design method for slope reliability evaluation by considering both stratigraphic uncertainty and spatial variability of soil parameters. Within the framework of full probability design, a generalized coupled Markov chain (GCMC) model is combined with a random field model to simultaneously characterize stratigraphic uncertainty and spatial variability of soil parameters. A slope is taken as an example for RBD using the borehole data in Perth, Australia.

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2 Stratigraphic Uncertainty Characterization Using GCMC Model

The GCMC model is developed on the basis of the traditional CMC model. This model not only considers the directional non-stationarity but also has a high computational efficiency (Park 2010).

The transition mode of generalized coupled Markov chains is from an unsampled cell to a sampled cell. In one-dimensional space, assuming the state of cell Z_1 to be estimated is S_i, the probability of transferring to element Z_2 with known state S_j is defined by Park (2010) as the transition likelihood $\pi_i(S_j)$. It can be written from Bayes′ theorem as

$$\pi_i(S_j) = \mathrm{P}(Z_2 = S_j \mid Z_1 = S_i) = \frac{\mathrm{P}(Z_1 = S_i, Z_2 = S_j)}{\mathrm{P}'(Z_1 = S_i)} \quad ,i = 1,\mathrm{K}\ ,n \tag{1}$$

where n represents total number of different soil states; Denominator $\mathrm{P}'(Z_1 = S_i)$ represents the prior probability of the soil state S_i. It can be determined by the ratio of each state in the previous stage and updated continuously during simulation. To consider directional nonstationarity, the numerator, $\mathrm{P}(Z_1 = S_i, Z_2 = S_j)$, is expressed as the transition probabilities for two sub-directions. The likelihood function is given by (Park 2010)

$$\pi_i(S_j) = \frac{\sqrt{\mathrm{P}(Z_2 = S_j \mid Z_1 = S_i)\mathrm{P}(Z_1 = S_i)\mathrm{P}(Z_1 = S_i \mid Z_2 = S_j)\mathrm{P}(Z_2 = S_j)}}{\mathrm{P}'(Z_1 = S_i)} \quad ,i = 1,\ldots,n \tag{2}$$

Markov chains often assume homogeneity, which indicates that the probability of soil state transition between adjacent soil units depends only on the relative location of soil cells, and has nothing to do with the absolute position of soil cells, i.e. $P(Z_i = S_w \mid Z_{i-1} = S_u) = P(Z_j = S_w \mid Z_{j-1} = S_u)$. The one-step transition probability matrices (TPMs) in two sub-directions can be represented by these two symbols, i.e. ${}^1p_{ij}$ and ${}^2p_{ij}$, respectively. The number 1 and 2 respectively denote the forward and backward direction. Using the r-step TPMs, Eq. (2) can be simplified to

$$\pi_i(S_j) = \frac{\sqrt{{}^1p_{ji}^{(d_1)} \cdot {}^2p_{ij}^{(d_1)} \cdot m_i \cdot m_j}}{m_i'}, \; i = 1,\ldots,n \tag{4}$$

where ${}^1p_{ji}^{(d_1)}$ and ${}^2p_{ij}^{(d_1)}$ are the components at (j, i) and (i, j) of the d_1 step TPMs, respectively. Where m_i , m_j and m_i' are represents $\mathrm{P}(Z_1 = S_i)$, $\mathrm{P}(Z_2 = S_j)$, and $\mathrm{P}'(Z_1 = S_i)$, respectively.

For a two-dimensional (2D) space, there are four sub-directional Markov chains, it is assumed that the sub-directional chain is independent of each other. The likelihood function is given by (Park 2010)

$$\pi_i(S_j) = \prod_{k=1}^{4} \frac{\sqrt{{}^k p_{ki}^{(d_k)} \cdot {}^{k^*} p_{ik}^{(d_k)} \cdot m_i \cdot m_{S(k)}}}{\mathrm{m}_i'} \Bigg/ \sum_{l=1}^{n} \prod_{k=1}^{4} \frac{\sqrt{{}^k p_{kl}^{(d_k)} \cdot {}^{k^*} p_{lk}^{(d_k)} \cdot m_l \cdot m_{S(k)}}}{\mathrm{m}_l'} \quad , i = 1,\ldots,n \tag{5}$$

where k^* is the opposite direction of k; $m_{S(k)}$ represents the marginal distribution of the corresponding state in the k-th sub-direction; the denominator is the correction factor ,ensuring summation of π_i is unity.

3 Spatial Variability Characterization of Soil Parameters Using Random Field Model

After the stratigraphic distribution is obtained by the method in the previous section, the soil types are mapped to the corresponding elements of the slope model. The slope profile will presents irregular distribution of various soil types, and its random field belongs to non-stationary random field. In this study, the spatial variability of soil parameters is simulated by the method mentioned in Lu and Zhang (2007). For a given soil profile, it can be divided into several non-overlapping sub-regions according to the soil type. The soil properties are assumed to be statistically stationary within each sub-region. The correlation coefficient between soil parameters at any two points in different region is assumed to be zero. For the same sub-region, the exponential autocorrelation function is adopted to characterize the autocorrelations between different points as follows:

$$\rho(\tau_x, \tau_y) = \exp\left[-2\left(\frac{\tau_x}{\delta_h} + \frac{\tau_y}{\delta_v}\right)\right] \tag{6}$$

where τ_x and τ_y represent the horizontal and vertical distance between two points respectively; δ_h and δ_v represent the horizontal and vertical scale of fluctuations (SOFs) of soil parameters, respectively.

The spatial variability of the parameters of soil shear strength in the same soil type area is characterized by a cross-correlated lognormal random field. The midpoint method based on Cholesky decomposition is used to discretize the above random fields. A cross-correlated lognormal random field can be expressed as follows (Deng et al. 2017):

$$\boldsymbol{H}_i\left(x,y\right)=\exp\left(\mu_{\ln i}+\sigma_{\ln i}\cdot\boldsymbol{H}_i^D\left(x,y\right)\right)\quad x,y\in\boldsymbol{\Omega}\,(\,i=c,\phi) \tag{7}$$

where $\sigma_{\ln i}=\sqrt{\ln\left(1+\left(\sigma_i/\mu_i\right)^2\right)}$ and $\mu_{\ln i}=\ln\mu_i-0.5\sigma_{\ln i}^2$; $\boldsymbol{H}_i^D$ is a cross-correlated Gaussian random field, which can be calculated by

$$\boldsymbol{H}_i^{\mathrm{D}}\left(x,y\right)=\boldsymbol{L}_2\cdot\boldsymbol{\chi}^{\mathrm{D}}=\boldsymbol{L}_2\cdot\boldsymbol{\xi}\cdot\boldsymbol{L}_1^{\mathrm{T}}\quad x,y\in\boldsymbol{\Omega}\,(\,i=c,\phi) \tag{8}$$

where $\boldsymbol{\xi}$ is an independent standard normal random sample matrix with size $N_e\times 2$, where N_e is the element number of random field; The cross-correlation coefficient matrix $\mathbf{R}_0$ can be decomposed by Cholesky to obtain the lower triangular matrix $\boldsymbol{L}_1$ with size 2×2, and the correlation standard normal random sample matrix $\boldsymbol{\chi}^{\mathrm{D}}$ can be obtained from $\boldsymbol{\xi}\cdot\boldsymbol{L}_1^{\mathrm{T}}$; $\boldsymbol{L}_2$ is another lower triangular matrix derived from Cholesky decomposition of autocorrelation coefficient matrix with size $N_e\times N_e$.

Stationary random field simulation is carried out in different sub-regions of a study area. After the corresponding random field is obtained, the distribution of shear strength parameters values corresponding to the whole region can be achieved.

4 Implementation Procedure

In order to consider both stratigraphic uncertainty and spatial variability of soil parameters into slope RBD, a full probabilistic design method combining GCMC model with random field model is proposed. The implementation procedure is presented, as shown in Figure 1.

5 Illustrative Examples

5.1 Borehole data

A set of borehole data from Central Business District, Perth, Western Australia is collected for analysis. The relative location of the borehole is shown in Figure 2(a), and Figure 2(b) gives the formation information reflected by each borehole. The *x*-axis is the relative position of borehole, and the *z*-axis is the drilling depth of borehole. As shown in Figure 1(b), the stratum mainly involves three soil types (i.e. sand, clay, and silt). One soil types is embedded in another and this stratum exhibits a certain degree of variability. According to the formation information reflected boreholes, the proportion of silt is relatively small, and the minimum thickness of geological unit is 0.4m. In order to conduct slope RBD, a slope is artificially constructed in this area, and its orientation is consistent with that of the red solid line in Figure 1(a). Table 1 summarizes the statistical characteristics of parameters of three soil types in the study area. Herein, only shear strength parameters are considered as random field variables, and other parameters are assumed to be constants. The SOFs of shear strength parameters in horizontal and vertical directions are assumed to be 40.0 m and 4.0 m respectively. The exponential correlation function is adopted in this study. The correlation coefficient between shear strength parameters is assumed to be -0.5, and the correlation coefficient between shear strength parameters of different soil types is assumed to be zero.

Table 1. Statistics of different types of soil parameters.

Soil types	Cohesion, *c*		Friction angle, *ϕ*		Unit weight, *γ*	Elastic Modulus, *E*	Poisson's ratio, *ν*
	Mean	COV	Mean	COV			
Sand	4 kPa	0.3	36°	0.2	20 kN·m^{-3}	50	0.3
Clay	32 kPa	0.3	25°	0.2	20 kN·m^{-3}	30	0.3
Silt	6 kPa	0.3	28°	0.2	20 kN·m^{-3}	30	0.3

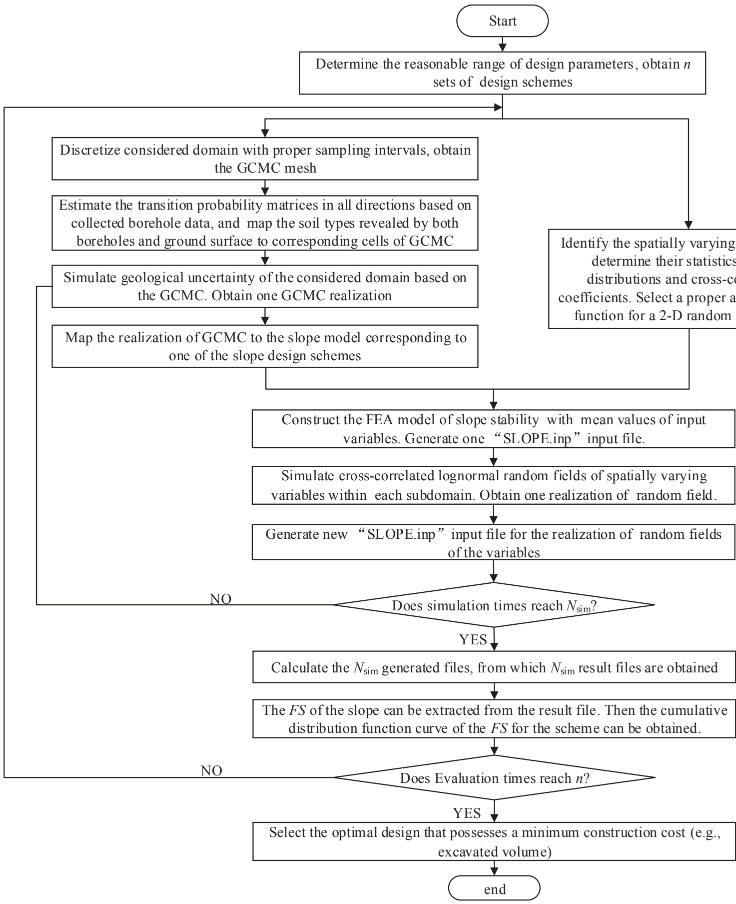
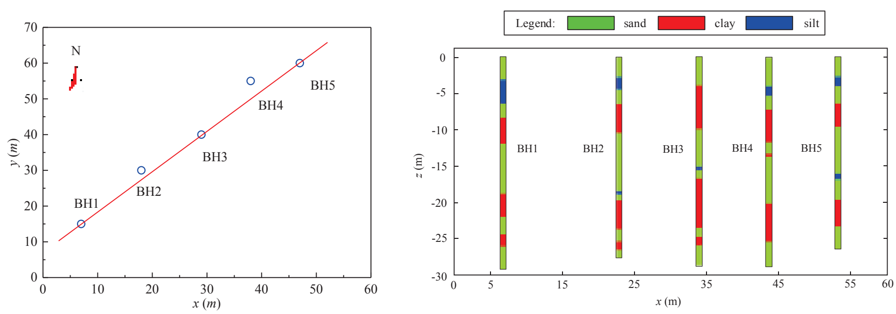


Figure 1. Flowchart of full probabilistic design method of slope considering the aforementioned two types of uncertainty.



(a) Relative locations of the boreholes

(b) Formation information reflected by each borehole

Figure 2. The borehole relative locations and formation information in Ireland used for analysis.

5.2 Slope calculation model

Before conduct the slope RBD, the parameters associated in the slope calculation model should be determined. The target probability of failure ($P_{f,\ T}$) level of the slope calculation model is set to 10^{-3} in this study. For simplicity, only the slope angle θ is taken as the design variable. The slope height fixes at 17.2m. The excavation quantity is used to measure the design cost C of the slope, the excavation volume of the slope is the excavation volume per unit length. The design space of the slope angle is divided into 12 values. Each value corresponds to a design scheme. The value ranges from 26° to 37°. The slope angle interval corresponding to each design scheme is 1°. The 12 design schemes are named D26, D27,…, and D37, where D26 represents the design scheme with slope angle 26°. To illustrate the importance of considering both stratum variability and soil parameter spatial variability in slope RBD, the results of slope reliability design considering only soil parameter spatial variability and both two kinds of variability are compared. The design scheme which only considers the spatial variability of soil parameters includes two cases: case a (i.e. the type of soil with weak shear strength in the stratum distribution used in the analysis accounts for a higher proportion) and case B (i.e. the type of soil with strong shear strength in the stratum distribution used in the analysis accounts for a higher proportion).

Firstly, the strata in the study area are meshed. The element size is 0.4m×0.8m and the total number of elements is 5625. The VTPMs in two vertical sub-directions can be obtained by statistical method based on collected five boreholes, which are summarized in Table 2. From the values in the table, it can be seen that the downward transition probability matrix (DTPM) and upward transition probability matrix (UTPM) are consistent, which indicates that the vertical downward and upward transfer in the study area remains stable. The HTPMs in two horizontal sub-directions, i.e. left-to-right transition probability matrix (LRTPM) and right-to-left transition probability matrix (RLTPM), can be estimated from Maximum likelihood estimation method based on borehole data. The estimated results of the HTPMs are shown in table 3. Through stratigraphic uncertainty simulation, the soil type distribution obtained from simulation is mapped to the slope model corresponding to each design scheme as known information, and the corresponding soil type distribution of slope can be obtained.

Table 2. VTPMs in two vertical sub-directions.

Direction		DTPM			UTPM	
Soil state	1 (clay)	2 (sand)	3 (silt)	1 (clay)	2 (sand)	3 (silt)
1 (clay)	0.897	0.069	0.034	0.897	0.069	0.034
2 (sand)	0.116	0.884	0.000	0.116	0.884	0.000
3 (silt)	0.318	0.000	0.682	0.318	0.000	0.682

Table 3. HTPMs in two vertical direction.

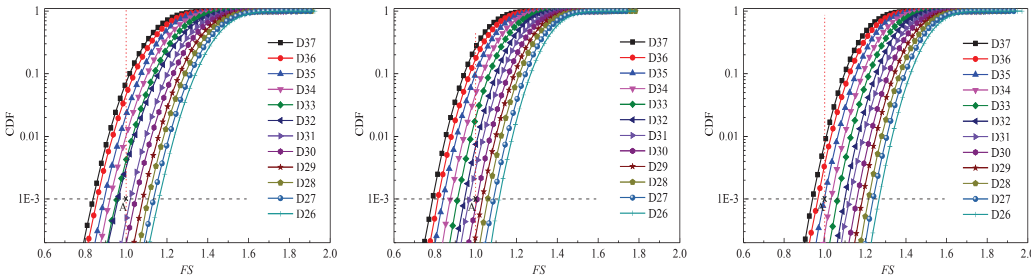
Direction		LRTPM			RLTPM	
Soil state	1 (clay)	2 (sand)	3 (silt)	1 (clay)	2 (sand)	3 (silt)
1 (clay)	0.969	0.021	0.010	0.967	0.022	0.011
2 (sand)	0.036	0.964	0.000	0.037	0.963	0.000
3 (silt)	0.117	0.000	0.883	0.120	0.000	0.880

5.3 Slope RBD results analysis

According to the full probability design method of slope reliability proposed in Section 4, cumulative distribution function (CDF) of *FS* for 12 schemes are given in Figure 3. Figure 3(a) represents the case in which both stratigraphic uncertainty and spatial variability of soil parameters are taken into account. Figure 3 (a) and (b) represent two cases (i.e. case *a* and *b*) in which only spatial variability of soil parameters is considered, respectively. Note that the number of Monte Carlo simulation for each design scheme is set as 1×10^4. The failure probability of each design scheme varies roughly in the order of 10^{-1} to 10^{-6}. If the slope failure probability is used to judge whether the design scheme achieves the target reliability, the Monte Carlo simulation times of each design scheme will be at least 1×10^7, and the total simulation times will be at least 12×10^7. The cost of this calculation will be enormous. To reduce the calculation cost, the relationship between CDF curve of *FS* and $P_{f,T}$ is used to judge whether the design scheme achieves the target reliability or not. If the *FS* value corresponding to the intersection point of the *FS* CDF curve and the $P_{f,T}$ of a design scheme is greater than 1, it means that the scheme satisfies the target reliability requirement, otherwise it is not satisfied. According to this method, the number of times needed for simulation is only related to the $P_{f,T}$. Since the $P_{f,T}$ is set at 10^{-3} in this study, the Monte Carlo simulation times of each design scheme are set at 1×10^4 to meet the convergence requirements. The total simulation times here are 12×10^4, which greatly reduces the cost of this calculation compared with the traditional methods.

The intersection point A of target failure probability and safety factor 1 can be found based on the *FS* CDF curve for each design scheme, see Figure 3(a). If the intersection point of the CDF curve and the straight line of $P_{f,T}=10^{-3}$ for a design scheme is on the right side of intersection point A, it shows that the design scheme meets the target reliability requirement, and vice versa, it indicates that the design scheme does not meet the target reliability requirement. As shown in Figure 3 (a), the design schemes D26, D27, D28, D29, D30, D31 and D32 meet the target reliability requirements. According to the design cost of each scheme, the optimal design scheme can be determined. Among them, D32 has the largest slope angle (i.e. the minimum excavation amount) in the design scheme which meets the reliability requirements of the target. It means D32 is the optimal design scheme.

According to the *FS* CDF curve for each design scheme in Figure 3(b), the corresponding intersection point A can also be found. Thus, it can be ascertained that the design schemes satisfying the target reliability are D26, D27, D28, D29, D30 and D31 when only considering the spatial variation of soil parameters. The design cost corresponding to D31 is the smallest, so the optimal design scheme for case *a* is D31. Scheme D31 is not the optimal design scheme although it satisfies the requirement of target reliability in the case of considering both stratigraphic uncertainty and spatial variability of soil parameters. This shows that the design scheme of case *a* will be conservative, which will increase the cost when only considering the spatial variation of soil parameters. Similarly, the intersection point A can be found from the *FS* CDF curve for the design schemes in Figure 3(c). From the Figure, it can be determined that the design schemes satisfying the target reliability only considering the spatial variation of soil parameters (case *b*) are D26, D27, D28, D29, D30, D31, D32, D33, D34 and D35, among which the design cost of D35 is the smallest, so the optimal design method corresponding to case *b* is D35. However, scheme D35 does not meet the reliability requirement of slope objective when considering both kinds of variability, which indicates that the optimal design scheme obtained by case *b* only considers the spatial variability of soil parameters will lead to dangerous slope design. Therefore, in order to obtain the optimal design scheme accurately, the influence of stratigraphic uncertainty and spatial variability of soil parameters should be taken into account in slope RBD.



(a) The case considering two types of variability

(b) The case considering only the spatial variation of soil parameters (case *a*)

(c) The case considering only the spatial variation of soil parameters (case *b*)

Figure 3. The corresponding CDF curves of various schemes for different cases.

6 Summary and Conclusion

This paper proposes a full probabilistic design method for the slope considering stratigraphic uncertainty and spatial variability of soil parameters. In the full probabilistic design framework, a GCMC model is combined with a random field model to simultaneously characterize stratigraphic uncertainty and spatial variability of soil parameters. A slope is taken as an example for RBD using the borehole data in Perth, Australia. The following conclusions are drawn from this study:

1. Within the framework of full probability design, a slope RBD method considering two types of variability of soil heterogeneity is proposed, which provides a more reasonable way for RBD.
2. If only spatial variability of soil parameters is considered in slope RBD, the design results largely depend on the distribution of the formation used by the engineer. In particular, when a stratigraphic distribution with a higher proportion of strong soil materials than the reality, the optimal design scheme will be at risk. In the opposite case, the optimal design scheme obtained will be conservative.
3. In order to obtain the optimal design scheme accurately, the both influences of stratigraphic uncertainty and spatial variability of soil parameters should be taken into account in slope RBD.

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