Study on autocorrelation model and reduction function of variance of soil random field

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ABSTRACT: The key to apply the random field theory to reliability is the confirmation of the reduction function of variance, which transforms the variable from 'point characteristic' to 'space characteristic'. A regular way is to choose the autocorrelation model from the experiment data, the reduction function of variance is deduced according to the autocorrelation function. The conventional reduction function of variance is modified based on the random field theory and CPT data of is an appropriate series for the analysis. A new autocorrelation model is brought forward and examples are analyzed to show the advantage of the new model, the results show that this model is adaptive with different autocorrelation function and the modified reduction function of variance is rational.

1 INTRODUCTION

Soil has prominent spatial variability because it has been formed by a combination of various geologic, environmental and physical-chemical processes, which results in soil properties varying vertically and horizontally (Phoon and Kulhawy, 1999a). Soil properties are modeled as 'random' because it's impossible to obtain measurements at all points, though soil are not 'random' in the sense that they are intrinsically unknown, so it is passive to establish random field for soil (Vanmarcke, 1983). Vanmarcke has researched random field theory systematically, dissertated the characteristic of soil random field incisively (Vanmarcke, 1977, 1983, 1986). Many people have studied on the sources of uncertainty in soil properties and its influence on design decisions (Phoon and Kulhawy, 1999a, 1999b; Cherubini, 2000). Analysis of how to choose the reduction function of variance reasonably have been studied extensively (YAN, 2006).

Spatial variability is an important factor affecting design. The spatial averaging of soil properties reduces its point variance (Sivakumar Babu, 2006). In the analysis, firstly, experimental autocorrelation function (ACF) for different lag distance is evaluated from the statistically homogenous data. Secondly, different autocorrelation models (ACM) are used to fit the ACF, and the correlation distance can be derived from the most appropriate ACM. Thirdly, the reduction function of variance is derived in terms of autocorrelation distance and averaging distance, the distance over which the geotechnical properties are averaged. This paper focuses on the reduction function of variance and the autocorrelation model. The reduction function of variance is modified based on the random field theory. The main two shape of ACF are bulgy and concave, but the existing models can only simulate one type of ACF, so a new ACM is provided, which can simulate both of them very well.

2 THEORY BACKGROUND

2.1 Numerical characteristic

The main numerical characteristics of random field are average function and autocorrelation function, suppose P is a point in the space, the random field may be expressed as $Y(P)$. If the average function is independent of coordinate, and the ACF only depends on the vector $\overline{PP'}$, the random field is homogeneous. In this paper, the soil random field is supposed to be normal homogeneous, which is expressed as *Y(z)*. The relation between *Y(z)* and *Y(P)* is described as:

$$Y(z) = Y(P) - \mu(z) \tag{1}$$

Where $\mu(z)$ is the average function.

2.2 Autocorrelation function

For one-dimensional normal homogeneous random field, defining ACF as:

$$\rho(\tau) = \frac{R(\tau)}{R(0)} \tag{2}$$

Where $R(0) = \sigma^2$.

2.3 Reduction function of variance

A way to deal with the spatial variability within the statistically homogeneous soil units (HSU) is through stochastic integral, which is defined as:

$$Y_h(z) = \frac{1}{h}\int_z^{z+h} Y(z)dz \tag{3}$$

In [z, z+h]. It can be proved that the average function of this stochastic integral is zero, the standard deviation δ_u may be calculated as:

$$Var[Y_h(z)] = \sigma^2[\frac{2}{h}\int_0^h (1-\frac{\tau}{h})\rho(\tau)d\tau] \tag{4}$$

The reduction function of variance can be defined as:

$$\Gamma^2(h) = \frac{2}{h}\int_0^h (1-\frac{\tau}{h})\rho(\tau)d\tau \tag{5}$$

This function describes the relationship between spatial deviation and point deviation. The reduction function of variance is a coefficient in [0,1], so the spatial deviation is less than the point deviation.

2.4 Correlation distance

Vanmarcke (1977) proposed that if the value of formula (6) existed δ_u would be the correlation distance of the random filed:

$$\lim_{h\to\infty} h\Gamma^2(h) = \delta_u \tag{6}$$

δ_u measures the distance within which the property shows relatively strong correlation or persistence from point to point. The values of the soil property within the distance are likely to be all above or all below the average. It is supposed that reduction function of variance can be described as:

$$\Gamma^2(h) = \begin{cases} 1 & (h \le \delta_u) \\ \delta_u / h & (h > \delta_u) \end{cases} \tag{7}$$

Regardless of the form of the underlying autocorrelation function, Vanmarcke (1983) modified the above equation as follows:

$$\Gamma^2(h) = \begin{cases} 1 & (h \le \frac{1}{2}\delta_u) \\ \frac{\delta_u}{h}\left(1-\frac{\delta_u}{4h}\right) & (h > \frac{1}{2}\delta_u) \end{cases} \tag{8}$$

3 MODIFIED OF REDUCTION FUNCTION OF VARIANCE

The key to application of random field theory to reliability analysis is to establish the reduction function of variance. Commonly, reduction function of variance is calculated by Eq.7, which supposes the autocorrelation model as follows:

$$\rho(\tau) = \begin{cases} 1 & (\tau \le \delta_u) \\ 0 & (\tau > \delta_u) \end{cases} \tag{9}$$

The reduction function of variance can be calculated by Eq.9 and Eq.5:

$$\Gamma^2(h)=\begin{cases}1 & (h\le\delta_u)\\ \frac{\delta_u}{h}+[\frac{\delta_u}{h}-(\frac{\delta_u}{h})^2] & (h>\delta_u)\end{cases} \tag{10}$$

Eq.9 can be replaced by Eq.7 appreciatively. But if the autocorrelation function is simulated by other autocorrelation model, Eq.7 would bring errors. For example, when the autocorrelation function is simulated by the single exponential model, $\rho(\tau)=e^{-b|\tau|}$, the reduction function of variance is given by:

$$\Gamma^2(h)=\frac{2}{b^2h^2}(bh+e^{-bh}-1) \tag{11}$$

It is easy to calculate $\delta_u=2/b$, so Eq. 11 can be expressed as:

$$\Gamma^2(h)=\frac{\delta_u}{h}+\frac{1}{2}(\frac{\delta_u}{h})^2(e^{-2h/\delta_u}-1) \tag{12}$$

When $h=\delta_u$, $\Gamma^2(h)=0.5677$. The standard deviations of the spatial averages would be lager than actual situation when using Eq.7, which results in the calculated reliability less than actual situation. Only when h is 'comparatively large', the Eq.7 can be used to calculated the reduction function of variance. But 'comparatively large' is a fuzzy concept, when on earth Eq.7 can be used accurately correspondingly? YAN shu-wang (2006) evaluated a concept, non-correlation distance (h^*), to solve this problem. When $h\le\delta_u$, $\Gamma^2(h)=1$; when $\delta_u<h\le h^*$, the reduction function of variance should be calculated by theoretic formula; when $h>h^*$, the reduction function of variance is constant, or calculated by δ_u/h. The disadvantage is this function is not continuous.

For different autocorrelation model, the reduction function of variance can be deduced as different equation based on the definition of reduction function of variance. Suppose when h is equal to h_k, the value of $\Gamma^2(h_k)$ calculated by Eq.7 closes to the value calculated by theoretic formula, the reduction function of variance can be modified as:

$$\Gamma^2(h)=\begin{cases}(\frac{\delta_u}{h_k})^{h/h_k} & (h\le h_k)\\ \delta_u/h & (h>h_k)\end{cases} \tag{13}$$

Where δ_u is autocorrelation distance, h_k can be evaluated from different autocorrelation model. The former is simulated as exponential function and the later is simulated as hyperbolic function.

4 NEW AUTOCORRELATION MODEL

The first thing to apply random field theory to reliability analysis is to choose the autocorrelation model based on the actual situation. Various kinds of autocorrelation model have been employed to fit the autocorrelation function (Vanmarcke, 1977; DeGroot, 1993; Fenton, 1999; and Phoon, 2003), the typical models are single exponential (SNX) and squared exponential (SQX). A new model is employed in this paper, which can be expressed as:

$$\rho(\tau)=\frac{1+a}{a+e^{b|\tau|}} \tag{14}$$

The reduction function of variance can be calculated as:

$$\begin{aligned}\Gamma^2(h)&=\frac{2}{h}\int_0^h(1-\frac{\tau}{h})\frac{1+a}{a+e^{b\tau}}d\tau\\ &=\frac{2(1+a)}{abh^2}[h\ln(1+a)-\int_0^h\ln(1+ae^{-b\tau})d\tau]\\ &=\frac{2(1+a)}{abh}[\ln(1+a)-\ln(1+ae^{-b\xi})]\quad \xi\in(0,h)\end{aligned} \tag{15}$$

Then the autocorrelation distance can be expressed as:

$$\delta_u=\lim_{h\to\infty}h\Gamma^2(h)=\frac{2(1+a)}{ab}\ln(1+a) \tag{16}$$

It's easy to calculate

$$\lim_{h\to 0}\Gamma^2(h)=\frac{2(1+a)}{ab}\lim_{h\to 0}\frac{h\ln(1+a)-\int_0^h \ln(1+ae^{-b\tau})d\tau}{h^2}$$
$$=1 \tag{17}$$
$$\lim_{h\to\infty}\Gamma^2(h)=\frac{2(1+a)}{ab}\lim_{h\to\infty}\frac{\ln(1+a)-\ln(1+ae^{-b\xi})}{h}$$
$$=0 \tag{18}$$

So Eq.14 can simulate the autocorrelation model, when a=0 this model changes to the SNX model. In the application of this model, first of all, the value of a, b can be calculated through function fitting, then the correlation distance can be calculated, the curves of Eq.15 and Eq.8 can be drawn to find hk , so the reduction function of variance can be attained from Eq.13. The analytical expression of the typical ACMs and the formulae relating the correlation distance to the model parameters are shown in Table 1.

Table1 different autocorrelation models and correlation distance

Autocorrelation model	Equation	Autocorrelation distance
SNX	$\rho(\tau)=\exp(-b\lvert\tau\rvert)$	$2/b$
SQX	$\rho(\tau)=\exp(-b^2\tau^2)$	$\sqrt{\pi}/b$
LGS	$\rho(\tau)=(1+a)/(a+\exp(b\lvert\tau\rvert))$	$2(1+a)\ln(1+a)/ab$

5 EXAMPLE

A good way to obtain Soil properties is through CPT data. Data of example 1 are collected from the USGS Earthquake Hazards Program (http://www.usgs.gov/); Data of example 2 are collected by the Georgia Tech In-Site Testing Group (http://geosystems.ce.gatech.edu/Faculty/Mayne/Research/index.html). An important prerequisite for reliability analyses is the identification of homogeneous soil units; M. UZIELLI (2005) introduced the identification of homogeneous of using CPT data.

The value of the soil property [$\xi(z)$] is constituted by a trend function [$t(z)$] and a fluctuating component $w(z)$, it can be expressed as:

$$\xi(z)=t(z)+w(z) \tag{19}$$

Where z is the depth coordinate.

It's important to remove the trend because it affects the correlation structure. The linear least-squares analysis is a good way to remove the trend. The *j*th coefficient of the sample autocorrelation function of the fluctuating component w_i can be calculated as follows

$$\rho(\tau_j)=\frac{\sum_{i=1}^{n_d-j} w_i\cdot w_{i+j}}{\sum_{i=1}^{n_d-j} w_i^2} \tag{20}$$

Where $\tau_j=j\Delta z, j=1,2,\cdots,n_d/4$, n_d is the number of data points, Δz is the sampling interval. A program, CACF, which can remove the trend with linear least-squares analysis and calculate the coefficient, was composed with Matlab to complete the calculation. Then the curve of $\rho(\tau)\sim\tau$ can be drawn, and the autocorrelation model would be chosen according to the trend of the curve. The curve should trend to zero as the distance increase, but the hinder part is usually anomalistic, it is possible because the calculation of $\rho(\tau)$ is not credible as τ increase. So it is reasonable to fit curve only to the initial part.

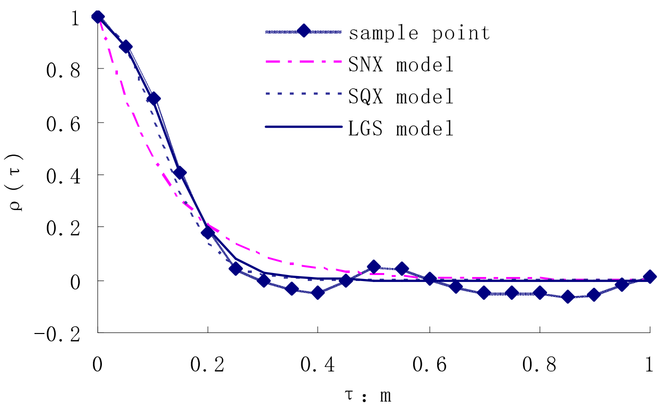

Fig.1 simulation of ACM of example 1

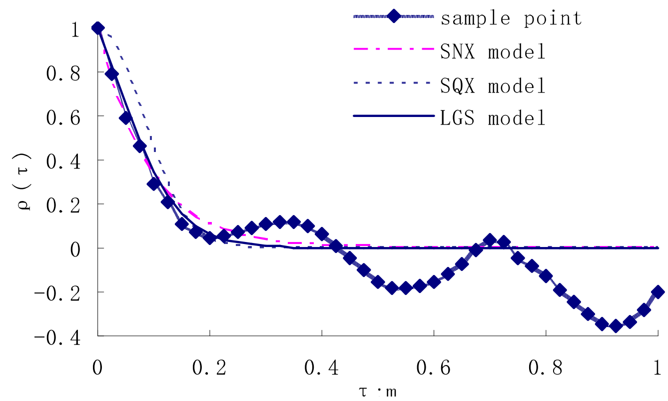

Fig.2 simulation of ACM of example 2

There are two kinds of autocorrelation function, one is the bulgy shape, the other is the concave shape, they may be observed in Fig.1 and Fig.2. From the figures we can see that the SQX can simulate the concave autocorrelation function, the SNX can simulate the bulgy autocorrelation function, and the model of this paper can simulate both kinds of autocorrelation function very well. Different autocorrelation models and correlation distance are shown in Table 2

Table2 different autocorrelation models and correlation distance

	SNX	SQX	LGS
Fig.1	0.250	0.278	0.280
Fig.2	0.178	0.184	0.173

After the correlation distance has been calculated, the reduction function of variance can be evaluated using Eq.13. For SNX, $h_k = 10\delta_u$, for SQX, $h_k = 4\delta_u$. Because there are two parameters in LGS, h_k can't be expressed with δ_u directly. After a and b have been calculated, the value of $\Gamma^2(h_k)$ can be calculated by Ea.15 with Matlab, then the curve of $h - \Gamma^2(h)$ and Eq.7 can be drawn in one figure, the value of h_k can be found in this figure. A program, CBCF, which can calculate h_k, was composed with Matlab. The value of h_k is 0.4m for example 1 and 0.32 for example 2. Put h_k into Eq.13, Reduction function of variance of example 1 can be expressed as:

$$\Gamma^2(h) = \begin{cases} (\frac{7}{10})^{h/0.4} & h \le 0.4 \\ 0.28/h & h > 0.4 \end{cases} \tag{21}$$

6 CONCLUSIONS

This paper modified the conventional expression of reduction function of variance based on the theory of random field. A new autocorrelation model, LGS, was evaluated. The results of the examples show that the model could simulate both kinds of autocorrelation function very well. And the modified expression of reduction function of variance was reasonable.

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REFERENCES

Phoon, K.K. & Kulhawy, F.H. (1999a). Characterization of geotechnical variability. Canadian geotechnical Journal, 36: 612-624

Vanmarcke, E.H. (1983). Random field: analysis and synthesis. MIT, Cambridge, Mass.

Vanmarcke, Erik H. (1977). Probabilistic Modeling of soil profiles. American Society of Civil Engineers, Journal of the Geotechnical Engineering Division, 103(11):1227-1246

Vanmarcke, Erik H. & Shinozuka M. et al. (1986). Random fields and stochastic finite element method. Structure Safety, 3: 143-168

Phoon, K.K. & Kulhawy, F.H. (1999b). Evaluation of geotechnical property variability. Canadian geotechnical Journal, 36: 625-639

Cherubini, C. (2000). Reliability evaluation of shallow foundation bearing capacity on c', Φ ' soils. Canadian geotechnical Journal, 37: 264-269

YAN Shu-wang & ZHU Hong-xia & LIU Run (2006). Study on application of random field theory to reliability analysis. Chinese Journal of Geotechnical Engineering, 28(12): 2053-2059

G.L. Sivakumar Babu, Amit Srivastava & D.S.N. Murthy (2006). Reliability analysis of the bearing capacity of a shallow foundation resting on cohesive soil. Canadian geotechnical Journal, 43: 217-223

DeGroot, D.J. &Baecher, G.B. (1993) Estimating autocovariances of in-situ soil properties. Journal of the Geotechnical Engineering, 119(1): 147-166

Fenton, G. (1999). Random field modeling of CPT data. J. Geotech. Geoenviron. Engng, 125(6): 486-498

Phoon, K.K, Quek, S. T. & An, P. (2003). Identification of statistically homogenous soil layers using modified Bartlett statistics, J. Geotech. Geoenviron. Engng, 129(7): 649-659

M. UZIELLI, G. VANNUCCHI, K.K. PHOON. Random field characterization of stress-normalized cone penetration testing parameters. Geotechnique, 2005, 55(1): 3-20